UNIVERSALITY FOR ONE–DIMENSIONAL HIERARCHICAL COALESCENCE PROCESSES WITH DOUBLE AND TRIPLE MERGES

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Abstract. We consider one–dimensional hierarchical coalescence processes (in short HCP) where two or three neighbouring domains can merge. An HCP consists of an infinite sequence of stochastic coalescence processes: each process occurs in a different "epoch" and evolves for an infinite time, while the evolutions in subsequent epochs are linked in such a way that the initial distribution of epoch $n+1$ coincides with the final distribution of epoch n. Inside each epoch a domain can incorporate one of its neighbouring domains or both of them if its length belongs to a certain epoch-dependent finite range.

Assuming that the distribution at the beginning of the first epoch is described by a renewal simple point process, we prove limit theorems for the domain length and for the position of the leftmost point (if any). Our analysis extends the results obtained in [FMRT0] to a larger family of models, including relevant examples from the physics literature [BDG], [SE]. It reveals the presence of a common abstract structure behind models which are apparently very different, thus leading to very similar limit theorems. Finally, we give here a full characterization of the infinitesimal generator for the dynamics inside each epoch, thus allowing to describe the time evolution of the expected value of regular observables in terms of an ordinary differential equation.

Keywords: coalescence process, simple point process, renewal process, universality, non-equilibrium dynamics.

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1. Introduction

A one-dimensional hierarchical coalescence process (HCP) consists of an infinite sequence of one-dimensional coalescence processes: each process occurs in a different *epoch* and evolves for an infinite time, while the evolution in subsequent epochs are linked in such a way that the initial distribution of epoch $n+1$ coincides with the final distribution of epoch n. At a given time inside epoch n the state of the process is described by a simple point process on $\mathbb{R}$, *i.e.* by a random locally finite subset of $\mathbb{R}$, such that the intervals among consecutive points (*domains*) are not smaller than $d^{(n)}$, where $\{d^{(n)}\}_{n\geq 1}$ is an a priori fixed sequence of strictly increasing and diverging positive numbers. The evolution inside epoch n can be informally described as follows. Only intervals whose length belongs to the finite range $[d^{(n)}, d^{(n+1)})$ are *active* *i.e.* they can incorporate their left neighbouring domain, their right neighbouring domain or both of them. Inactive domains cannot incorporate their neighbours and can increase their length only if they are incorporated by active neighbours. The rates of the merging events and the sequence $\{d^{(n)}\}_{n\geq 1}$ are quite general, with the

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important feature that the activity ranges $[d^{(n)}, d^{(n+1)})$ should be such that after each merging step the newly produced interval always becomes inactive for that epoch but active for some future epoch.

We have introduced the concept of HCP in [FMRT0], considering only left or right merging of domains, *i.e.* a domain cannot incorporate simultaneously both its neighbours. There we proved that if the initial distribution is a renewal process, such property is preserved at all times and epochs and the distribution of certain rescaled variables – the domain length and the position of the leftmost point (if any) – has a well defined limiting behaviour corresponding to large universality classes (most of the dynamical details disappear in the scaling limit). Here we extend these results to the more general HCP's defined above which also allow triple merging and we determine the corresponding limiting behavior and universality classes.

Besides the mathematical interest, our study has been motivated by the fact that several HCP's have been implicitely introduced in physics literature to model the non-equilibrium evolution of one dimensional systems whose dynamics is dominated by the coalescence of proper domains or droplets characterizing the experiments. We refer to Section 2.4 for a review of some of these HCP's and the corresponding physical systems. A key common feature emerges from the experiments on all these systems: an interesting coarsening phenomena occurs which leads to a scale-invariant morphology for large times, namely the system is described by a single (time-dependent) length and the distribution approaches a scaling form. Several models, even very simple ones, have been proposed by physicists in order to capture and explain such intriguing behavior and in many cases these models turn out to be HCP's (see *e.g.* [P], [DBG], [DGY1], [DGY2], [SE], [BDG]). Supported by computer simulations and under the key assumption of a well defined limiting behavior under suitable rescaling, physicists have derived for these HCP's in the mean field approximation some non trivial limiting distributions for the relevant quantities and noticed that these distributions display a certain degree of universality. The results we obtained in [FMRT0] prove and generalize the findings of physicists. However the analysis in [FMRT0] does not cover some cases of interests for physics which involve triple merging, *e.g.* the HCP which has introduced in [BDG] to model Ising at zero temperature (see Section 2.4.3). These models are instead covered by the present study which explains why the limiting distributions of several models, although different, have a similar structure.

The analysis in [FMRT0] is based on a robust combinatorial study of the coalescence inside a given epoch, which becomes extremely hard in the present setting. Hence, here we have followed a different route inspired by the approach of [SE]. In particular, we start with the infinitesimal generator of the one–epoch coalescence, giving a complete characterization of its form and domain (Theorem 2.9). As well known, this allows to characterize the time evolution of the expectation of regular observables in terms of an ordinary differential equation. Applied to the domain length and the position of the leftmost point (if any), this method leads to recursive equations between the Laplace transforms of the involved quantities at the beginning and the end of each epoch, and therefore at the beginning of two consecutive epochs (Theorems 2.6 and 2.8).

The study of the Markov generator for stochastic processes whose state at a give time is described by a simple point process on $\mathbb{R}$ is rather heavy [Pr]. Here, we have introduced a lattice structure (which is somehow artificial from a geometric

point of view) that strongly simplifies the analysis of the Markov generator, and in particular allows us to use the standard methods described in [L]. However, such a discretization requires some very special care, because of the use the vague topology on the the space $\mathcal{N}$ of locally finite subset of $\mathbb{R}$. Once obtained the above mentioned system of recursive equations between Laplace transforms, we have generalized the transformation introduced in [FMRT0, Section 5] which in some sense linearizes the system and allows to analyze the recursive identities and obtain the limit behavior (Theorems 2.12 and 2.15). The resulting transformation is now a more abstract object and can therefore be applied to a larger class of models.

Finally we stress that the heuristic technique developed by physicists (see [BDG]) to derive the limiting distribution (under the assumption of the existence of a limiting behaviour) is restricted to models with $d^{(n)} = n$ and it becomes meaningless also at heuristic level if the ratio $d^{(n)}/d^{(n+1)}$ does not converge to 1 as n goes to ∞. Under the same hypothesis of [BDG], namely $d^{(n)} = n$ and via the mean field approximation, in [CP] the authors proposed a time evolution equation which should describe the domain size distribution when the time variable t is a continuous approximation of the discrete label n of the epochs and one forgets how much time elapses between and during the merging events. This equation has been rigorously analyzed in [CP] and [GM] and in the latter work a limiting self-similar profile for this equation has been proved. In this special case, a transformation similar to one presented in more generality in [FMRT0], and here, has been used.

2. Model and results

In this section we fix some notation and give our main results. We first introduce the simple point processes we are interested in (standard references are [DV], [FKAS]). Then we define the process called one–epoch coalescence process (in short OCP) and the hierarchical coalescence process (HCP). Finally we provide some examples of HCP's coming from the physics literature.

2.1. **Simple point processes (SPP).**

We denote by $\mathcal{N}$ the family of locally finite subsets $\xi \subset \mathbb{R}$. $\mathcal{N}$ is a measurable space endowed with the σ–algebra of measurable subsets generated by

$$\{\xi \in \mathcal{N} \,:\, |\xi \cap A_1| = n_1 \,, \cdots , |\xi \cap A_k| = n_k\},$$

$A_1, \dots, A_k$ being bounded Borel sets in $\mathbb{R}$ and $n_1, \dots, n_k \in \mathbb{N}$. We recall that any probability measure on the measurable space $\mathcal{N}$ defines a simple point process (SPP).

We call *domains* the intervals $[x, x']$ between nearest–neighbour points x, x' in $\xi \cup \{-\infty, +\infty\}$. Note that the existence of the domain $[-\infty, x']$ corresponds to the fact that ξ is bounded from the left and its leftmost point is given by x'. A similar consideration holds for $[x, \infty]$. Points of ξ are also called *domain separation points*. Given a point $x \in \mathbb{R}$, we define

$$d_x^\ell := \inf\{t > 0 \,:\, x - t \in \xi\} \,, \qquad d_x^r := \inf\{t > 0 \,:\, x + t \in \xi\},$$

with the convention that the infimum of the empty set is ∞. Note that if $x \in \xi$ then d_x^ℓ (d_x^r) is simply the length of the domain to the left (right) of x.

In what follows $\mathbb{N}$ ($\mathbb{N}_+$) will denote the set of non-negative (positive) integers.

Definition 2.1.

(i) We say that a SPP ξ is left-bounded *if it has a leftmost point and has infinite cardinality.*
(ii) We say that a SPP ξ is $\mathbb{Z}$–stationary *if $\xi \subset \mathbb{Z}$ and its law $\mathcal{Q}$ is invariant by $\mathbb{Z}$–translations, i.e. if for any $x \in \mathbb{Z}$ the random set $\xi - x$ has law $\mathcal{Q}$.*
(iii) We say that a SPP ξ is stationary *if its law $\mathcal{Q}$ is invariant under $\mathbb{R}$-translations, i.e. if for any $x \in \mathbb{R}$ the random set $\xi - x$ has law $\mathcal{Q}$.*

If ξ is $\mathbb{Z}$–stationary or stationary, then a.s. the following dichotomy holds [FKAS]: ξ is unbounded from the left and from the right or ξ is empty. In the sequel we will always assume the first alternative to hold a.s. and we will write $\xi = \{x_k : k \in \mathbb{Z}\}$ with the rules: $x_0 \leq 0 < x_1$ and $x_k < x_{k+1}$ for all $k \in \mathbb{Z}$. In the case of a left–bounded SPP, we enumerate the points of ξ as $\{x_k : \ k \in \mathbb{N}\}$ in increasing order.

We now describe the main classes of SPP's we are interested in.

Definition 2.2. *Let ν and μ be probability measures on $\mathbb{R}$ and $(0, \infty)$, respectively. Let ξ be a SPP with law $\mathcal{Q}$.*

- *We say that ξ is a* renewal SPP containing the origin and with interval law μ*, and write $\mathcal{Q} = Ren(\mu \,|\, 0)$, if*
  - *(i) $0 \in \xi$,*
  - *(ii) ξ is unbounded from the left and from the right and, labelling the points in increasing order with $x_0 = 0$, the random variables $d_k = x_k - x_{k-1}$, $k \in \mathbb{Z}$, are i.i.d. with common law μ.*
- *We say that ξ is a* right renewal *SPP with first point law ν and interval law μ, and write $\mathcal{Q} = Ren(\nu, \mu)$, if*
  - *(i) $\xi = \{x_k\,, k \in \mathbb{N}\}$ is a left-bounded SPP,*
  - *(ii) the first point x_0 has law ν,*
  - *(iii) $d_k = x_k - x_{k-1}$ ($k \in \mathbb{N}_+$) has law μ,*
  - *(iv) the random variables $x_0, \{d_k\}_{k \in \mathbb{N}_+}$ are independent.*
- *If μ has finite mean, we say that ξ is a stationary renewal SPP with interval law μ, and write $\mathcal{Q} = Ren(\mu)$, if*
  - *(i) ξ is a stationary SPP with finite intensity and ξ is non-empty a.s.,*
  - *(ii) the random variables $d_k = x_k - x_{k-1}$, $k \in \mathbb{Z}$, are i.i.d. with common law μ w.r.t. the Palm distribution associated to $\mathcal{Q}$.*
- *If μ has support on $\mathbb{N}_+$ and has finite mean, we say that ξ is a $\mathbb{Z}$–stationary* renewal SPP with interval law *μ, and write $\mathcal{Q} = Ren_{\mathbb{Z}}(\mu)$, if*
  - *(i) ξ is $\mathbb{Z}$–stationary and a.s. non-empty,*
  - *(ii) w.r.t. the conditional probability $\mathcal{Q}(\cdot | 0 \in \xi)$ the random variables $d_k = x_k - x_{k-1}$, $k \in \mathbb{Z}$, are i.i.d. with common law μ.*

We recall that the intensity $\lambda_{\mathcal{Q}}$ of a stationary SPP with law $\mathcal{Q}$ is defined as the expectation $\lambda_{\mathcal{Q}} := \mathbb{E}_{\mathcal{Q}}\left(|\xi \cap [0,1]|\right)$. A ($\mathbb{Z}$–)stationary renewal SPP with interval law μ having infinite mean cannot exist (see Proposition 4.2.I in [DV] and Appendix C in [FMRT0]). As discussed after Theorem 1.3.4 in [FKAS], $\mathcal{Q} = \mathrm{Ren}(\mu)$ if and only if the following holds: the random variables $d_k = x_k - x_{k-1}$, $k \neq 1$, are i.i.d. with law μ and are independent from the random vector (x_0, x_1), which satisfies

$$\mathcal{Q}(-x_0 > u,\, x_1 > v) = \lambda_{\mathcal{Q}} \int_{u+v}^{\infty} \big(1 - F(t)\big)\, dt\,, \quad F(t) := \mu\big((0,t]\big)\,, \qquad u, v > 0. \tag{1}$$

2.2. **The one–epoch coalescence process (OCP).** This process depends on two constants $0 < d_{\min} < d_{\max}$ and on non-negative bounded continuous functions

$\lambda_\ell, \lambda_r, \lambda_a$ defined on $[d_{\min}, \infty]$ which, with $\lambda(d) := \lambda_\ell(d) + \lambda_r(d) + \lambda_a(d)$, satisfy the following assumptions:

(A1) $\lambda(d) > 0$ if and only if $d \in [d_{\min}, d_{\max})$,
(A2) if $d, d' \geq d_{\min}$, then $d + d' \geq d_{\max}$.

Trivially, (A2) is equivalent to the bound $2d_{\min} \geq d_{\max}$.

The admissible starting configurations for the OCP belong to the subset $\mathcal{N}(d_{\min})$ given by the configurations $\xi \in \mathcal{N}$ having only domains of length not smaller than $d_{\min}$, *i.e.*

$$\mathcal{N}(d_{\min}) = \{\xi \in \mathcal{N} \,:\, d_x^\ell \geq d_{\min}\,,\; d_x^r \geq d_{\min} \;\forall x \in \xi\}. \tag{2}$$

Then, the stochastic evolution of the OCP is given by a jump dynamics with càdlàg paths $\{\xi(t)\}_{t\geq 0}$ in the Skorohod space $D\big([0,\infty), \mathcal{N}(d_{\min})\big)$ (cf. [B]). Roughly speaking, the dynamics is the following. Each domain Δ of length d waits an exponential time with parameter $\lambda(d)$, afterwards exactly one of the following annihilations takes place: the left extreme of Δ is erased with probability $\lambda_\ell(d)/\lambda(d)$, the right extreme of Δ is erased with probability $\lambda_r(d)/\lambda(d)$, both the extremes of Δ are erased with probability $\lambda_a(d)/\lambda(d)$. We say that the domain Δ incorporates its left domain, its right domain, both its neighbouring domains, respectively. In Section 8 we present a full construction of all OCPs, varying the initial configuration, on the same probability space (*universal coupling*).

Note that the assumptions (A1) and (A2) on the coalescence rates imply that any domain which has been generated by a coalescence event is not active, *i.e.* it cannot incorporate other domains. This assumption comes from several models of physical interest (see Section 2.4) and plays a fundamental role in our analysis.

Remark 2.3. *Note that λ_ℓ and λ_r correspond to λ_r^* and λ_ℓ^* in* [FMRT0]*. The case $\lambda_a \equiv 0$ has been treated in* [FMRT0] *without the additional assumption that λ_ℓ, λ_r are continuous functions.*

Formally, the Markov generator of the OCP is given by

$$\mathcal{L}f(\xi) = \sum_{\substack{[x,x+d]\\ \text{domain in } \xi}} \Big\{ \lambda_\ell(d)\big[f(\xi \setminus \{x\}) - f(\xi)\big] + \lambda_r(d)\big[f(\xi \setminus \{x+d\}) - f(\xi)\big] + \lambda_a(d)\big[f(\xi \setminus \{x, x+d\}) - f(\xi)\big] \Big\}. \tag{3}$$

A precise description of the Markov generator $\mathcal{L}$ is given below while its full rigorous analysis is postponed to Section for clarity of exposition.

We will write $\mathbb{P}_{\mathcal{Q}}$ for the law on $D\big([0,\infty), \mathcal{N}(d_{\min})\big)$ of the OCP with initial law $\mathcal{Q}$ on $\mathcal{N}(d_{\min})$ and $\mathcal{Q}_t$ for its marginal at time t.

Since the OCP is an annihilation process, points can only disappear. Furthermore, Assumptions $(A1)$ and $(A2)$ guarantee that the process converges to a limiting configuration. One can easily prove the following lemma already stated in [FMRT0] in a less general setting (details are left to the reader).

Lemma 2.4. *For any given initial condition $\xi \in \mathcal{N}(d_{\min})$ the following holds:*

(i) $\xi(t) \subset \xi(s)$ *if* $s \leq t$,
(ii) *there exists a unique element $\xi(\infty)$ in $\mathcal{N}(d_{\max})$ such that $\xi(t) \cap I = \xi(\infty) \cap I$ for all large enough t (depending on I) and all bounded intervals I.*

The next result is a simple generalization of [FMRT0, Theorem 2.13] (its proof is based on the universal coupling described in Section 8, we omit details). It states that if the process starts with some right renewal (respectively stationary, $\mathbb{Z}$–stationary, etc.) simple point process ξ, then at any later time t, the process $\xi(t)$ is still of the same type.

Lemma 2.5. *Let ν, μ be two probability measures on $\mathbb{R}$ and $[d_{\min}, \infty)$, respectively. Then, for all $t \in [0, \infty]$ there exist probability measures ν_t, μ_t on $\mathbb{R}$ and $[d_{\min}, \infty)$ respectively such that $\nu_0 = \nu$, $\mu_0 = \mu$ and*

(i) if $\mathcal{Q} = Ren(\nu, \mu)$ then $\mathcal{Q}_t = Ren(\nu_t, \mu_t)$,
(ii) if $\mathcal{Q} = Ren(\mu)$ then $\mathcal{Q}_t = Ren(\mu_t)$,
(iii) if $\mathcal{Q} = Ren_{\mathbb{Z}}(\mu)$ then $\mathcal{Q}_t = Ren_{\mathbb{Z}}(\mu_t)$,
(iv) if $\mathcal{Q} = Ren(\delta_0, \mu)$ then $\mathcal{Q}_t(\cdot \,|\, 0 \in \xi) = Ren(\delta_0, \mu_t)$,
(v) $\lim_{t\to\infty} \nu_t = \nu_\infty$ and $\lim_{t\to\infty} \mu_t = \mu_\infty$ weakly.

Thanks to the previous results $\xi(\infty)$, μ_∞ and ν_∞ are well defined. In fact there exists a recursive identity between the Laplace transform of the interval law, and of the first point law, at time $t = 0$ and at time $t = \infty$. These identities, stated in Theorem 2.6 and Theorem 2.8 below, will be the keystones of the analysis of the asymptotic of the hierarchical coalescence process.

Given a probability measures μ on $[d_{\min}, \infty)$, let μ_t be as in Lemma 2.5. Then, for $s \in \mathbb{R}_+$, define

$$G_t(s) = \int e^{-sx} \mu_t(dx) \,, \qquad H_t(s) = \int_{[d_{\min}, d_{\max})} e^{-sx} \mu_t(dx).$$

Theorem 2.6 (Recursive identities for the interval law)**.** *For any $s \in \mathbb{R}_+$, the functions $[0, \infty) \ni t \mapsto G_t(s), H_t(s)$ are differentiable and satisfy*

$$\partial_t H_t(s) = -\int \mu_t(dx)\lambda(x)e^{-sx}, \tag{4}$$

$$\partial_t \big[G_t(s) - H_t(s)\big] = G_t(s) \int \mu_t(dx)(\lambda_\ell + \lambda_r)(x)e^{-sx} + G_t(s)^2 \int \mu_t(dx)\lambda_a(x)e^{-sx}. \tag{5}$$

In particular, it holds

(i) If $\lambda_a \equiv 0$, then $\partial_t G_t(s) = \partial_t H_t(s)(1 - G_t(s))$. Hence,

$$1 - G_t(s) = (1 - G_0(s))e^{H_0(s) - H_t(s)}, \qquad t \in \mathbb{R}_+, \tag{6}$$

$$1 - G_\infty(s) = (1 - G_0(s))e^{H_0(s)}. \tag{7}$$

(ii) If $\lambda_\ell + \lambda_r \equiv \gamma\lambda_a$ for some $\gamma \geq 0$, then $\partial_t G_t(s) = \partial_t H_t(s) \Big(1 - \frac{G_t(s)(\gamma + G_t(s))}{1+\gamma}\Big)$. Hence, for $s > 0$ it holds

$$e^{-\frac{\gamma+2}{\gamma+1} H_t(s)} \frac{\gamma + 1 + G_t(s)}{1 - G_t(s)} = e^{-\frac{\gamma+2}{\gamma+1} H_0(s)} \frac{\gamma + 1 + G_0(s)}{1 - G_0(s)}, \qquad t \in \mathbb{R}_+, \tag{8}$$

$$\frac{\gamma + 1 + G_\infty(s)}{1 - G_\infty(s)} = e^{-\frac{\gamma+2}{\gamma+1} H_0(s)} \frac{\gamma + 1 + G_0(s)}{1 - G_0(s)}. \tag{9}$$

In the above theorem, as in the rest of the paper, differentiability at $t = 0$ for a function on $[0, \infty)$ means differentiability from the right.

Remark 2.7. *The restriction to the above cases (i) and (ii) is technical and motivated by the following. Set*

$$a_t(s) := \int \mu_t(dx)(\lambda_\ell + \lambda_r)(x)e^{-sx} \quad \text{and} \quad b_t(s) := \int \mu_t(dx)\lambda_a(x)e^{-sx}.$$

Thanks to (4)*, Equation* (5) *can be rewritten as*

$$\partial_t G_t(s) = -a_t(s) - b_t(s) + a_t(s)G_t(s) + b_t(s)G_t(s)^2. \tag{10}$$

Fixing functions $A_t(s)$ and $B_t(s)$ such that $\partial_t A_t(s) = a_t(s)$ and $\partial_t B_t(s) = b_t(s)$, (10) *leads to*

$$\frac{e^{A_t(s)+2B_t(s)}}{1-G_t(s)} = \frac{e^{A_0(s)+2B_0(s)}}{1-G_0(s)} + \int_0^t b_u(s)e^{A_u(s)+2B_u(s)}du. \tag{11}$$

In order to have a recursive identity between (G_0, H_0) and G_∞, one needs to find an explicit expression of the integral in the right hand side of (11)*. This can be achieved in cases (i) and (ii) of Theorem 2.6 by taking $B_t(s) = b_t(s) = 0$ and $A_t(s) = -H_t(s)$ in case (i), and by taking $A_t(s) = \gamma B_t(s)$ and $A_t(s) + B_t(s) = -H_t(s)$ in case (ii).*

Finally we point out that, since $\operatorname{arctanh}(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$ *for $x \in (-1,1)$,* (8) *with $\gamma = 0$ can be rewritten in the more compact form*

$$-H_t + \operatorname{arctanh} G_t(s) = -H_0 + \operatorname{arctanh} G_0(s).$$

The next result is concerned with the evolution of the first point law ν_t when starting with a SPP having law $\mathrm{Ren}(\nu, \mu)$ (recall Lemma 2.5). First, we observe that if ξ is a SPP with law $\mathrm{Ren}(\delta_0, \mu)$ and V is a random variable with law ν independent from ξ, then the translated random subset $\{x + V : x \in \xi\} \subset \mathbb{R}$ is a SPP with law $\mathrm{Ren}(\nu, \mu)$. This simple observation and the definition of the OCP, whose dynamics depends only on the sequence of the domain lengths and not on the specific location of the domains, allow to conclude that ν_t is the convolution

$$\nu_t = \bar{\nu}_t * \nu, \tag{12}$$

where $\bar{\nu}_t$ denotes the evolution at time t of the first point law when starting from a SPP having law $\mathrm{Ren}(\delta_0, \mu)$. Hence, without loss we can restrict our analysis to this case.

Theorem 2.8 (Recursive identities for the first point law)**.** *Assume that $\nu = \delta_0$. Then, for any $s \in \mathbb{R}_+$ the Laplace transform*

$$[0,\infty) \ni t \mapsto L_t(s) := \int e^{-sx}\nu_t(x) \in (0,1]$$

is differentiable and satisfies

$$\frac{\partial_t L_t(s)}{L_t(s)} = -\int \mu_t(dy)\left(\lambda_\ell(y) + \lambda_a(y)\right) + \int \mu_t(dy)\lambda_\ell(y)e^{-sy} + G_t(s)\int \mu_t(dy)\lambda_a(y)e^{-sy}. \tag{13}$$

In particular, it holds:

(i) If $\lambda_a \equiv 0$ and $\lambda_r \equiv \gamma\lambda_\ell$ for some constant $\gamma \geq 0$, then it holds $\partial_t L_t(s) = \frac{L_t(s)}{1+\gamma}\left(\partial_t H_t(0) - \partial_t H_t(s)\right)$. Hence,

$$L_t(s) = L_0(s)\exp\left\{\frac{-H_t(s) + H_t(0) + H_0(s) - H_0(0)}{1+\gamma}\right\}, \qquad t \in \mathbb{R}_+, \tag{14}$$

$$L_\infty(s) = L_0(s)\exp\left\{\frac{H_0(s) - H_0(0)}{1+\gamma}\right\}. \tag{15}$$

If $\lambda_a \equiv 0$ and $\lambda_\ell \equiv 0$, then trivially $L_t(s) = L_0(s)$ for any $t \geq 0$.

(ii) *If $\lambda_\ell \equiv 0$ and $\lambda_r \equiv 0$, then $\partial_t L_t(s) = L_t(s)(\partial_t H_t(0) - G_t(s)\partial_t H_t(s))$. Hence, for $s > 0$ it holds*

$$L_t(s) = L_0(s)\sqrt{\frac{1-G_t^2(s)}{1-G_0^2(s)}}e^{H_t(0)-H_0(0)}, \qquad t \in \mathbb{R}_+, \tag{16}$$

$$L_\infty(s) = L_0(s)\sqrt{\frac{1-G_\infty^2(s)}{1-G_0^2(s)}}e^{-H_0(0)}. \tag{17}$$

We point out that cases (i) and (ii) of Theorem 2.8 are included into (but not equal to) cases (i) and (ii) of Theorem 2.6.

The previous results are based on our analysis of the Markov generator $\mathcal{L}$ of the OCP. In general, the expected value at time t of a regular observable evolves according to an ordinary differential equation that we describe below. We first fix some notation. Given $k \in \mathbb{Z}$ we set

$$I_k := \begin{cases} [kd_{\min}, (k+1)d_{\min}) & \text{if } k \geq 1, \\ (kd_{\min}, (k+1)d_{\min}) & \text{if } k = 0, \\ (kd_{\min}, (k+1)d_{\min}] & \text{if } k \leq -1. \end{cases} \tag{18}$$

Given $\xi \in \mathcal{N}(d_{\min})$, we set for $k \in \mathbb{Z}$ and $k < k'$ in $\mathbb{Z}$:

$$\begin{aligned} \xi^k &:= \xi \setminus I_k & \nabla_k f(\xi) &:= f(\xi^k) - f(\xi), \\ \xi^{k,k'} &:= \xi \setminus (I_k \cup I_{k'}) & \nabla_{k,k'} f(\xi) &:= f(\xi^{k,k'}) - f(\xi). \end{aligned}$$

We define

$$\mathcal{R} := \mathbb{Z} \cup \{(k,k') \,:\, k' \in \{k+1, \dots, k + \lceil d_{\max}/d_{\min} \rceil\}, k, k' \text{ in } \mathbb{Z}\},$$

$\lceil a \rceil$ being the smaller integer $n \geq a$. We consider the space $\mathcal{N}(d_{\min})$ endowed of the vague topology (see Section 3), making it a compact space. We write $\mathbb{B}$ for the Banach space of all continuous functions $f : \mathcal{N}(d_{\min}) \mapsto \mathbb{R}$ endowed with the uniform norm that we denote by $\|\cdot\|$. Also, and for later purpose, we let $\mathbb{B}_{\text{loc}}$ be the set of functions $f \in \mathbb{B}$ that are local, *i.e.* such that there exists a bounded interval $I \subset \mathbb{R}$ with $f(\xi) = f(\xi \cap I)$ for all $\xi \in \mathcal{N}(d_{\min})$. Then, similarly to the analysis of interacting particle systems [L], we define

$$\Delta_f(r) := \sup_{\xi \in \mathcal{N}(d_{\min})} \left|\nabla_r f(\xi)\right|, \quad f \in \mathbb{B}, r \in \mathcal{R}.$$

and we introduce the subset $\mathbb{D}$ of $\mathbb{B}$ as

$$\mathbb{D} := \left\{f \in \mathbb{B} \,:\, |||f||| := \sum_{r \in \mathcal{R}} \Delta_f(r) < \infty\right\}. \tag{19}$$

Observe that $\mathbb{B}_{\text{loc}} \subset \mathbb{D}$. The following result characterize completely the Markov generator of the OCP:

Theorem 2.9. *The subspaces $\mathbb{B}_{\text{loc}}$ and $\mathbb{D}$ are a core of the Markov generator $\mathcal{L}$, i.e. $\mathcal{L}$ is the closure of the operator obtained by restriction to $\mathbb{B}_{\text{loc}}$ or to $\mathbb{D}$. Moreover, if $f \in \mathbb{D}$, $\mathcal{L}f(\xi)$ equals the absolutely convergent series in the r.h.s. of* (3).

The proof is given in Section 9. Although this analysis represents our starting point, we prefer to postpone it to the end since rather technical. As a consequence of

the above theorem and standard theory of Markov generators, we get the following characterization of the time evolution of expected observables:

Corollary 2.10. *Given $f \in \mathbb{D}$, the the map $f(t,\xi) := \mathbb{E}_\xi\big[f(\xi_t)\big]$ (the expectation of f for the OCP at time t starting from ξ) is differentiable in t as function in $\mathbb{B}$ and moreover $\frac{d}{dt} f(t,\cdot) = \mathcal{L} f$.*

2.3. **The hierarchical coalescence process.** We can now introduce the *hierarchical coalescence process* (in short HCP). The dynamics depends on a strictly increasing sequence of positive numbers $\{d^{(n)}\}_{n\geq 1}$ and a family of bounded continuous functions $\lambda_\ell^{(n)}$, $\lambda_r^{(n)}$, $\lambda_a^{(n)} : [d^{(n)}, \infty] \to [0, A_n]\,, n \geq 1$. Without loss of generality, at cost of a length rescaling, we may assume

$$d^{(1)} = 1. \tag{20}$$

We set $\lambda^{(n)} := \lambda_\ell^{(n)} + \lambda_r^{(n)} + \lambda_a^{(n)}$ and we assume

(A1) for any $n \in \mathbb{N}_+$, $\lambda^{(n)}(d) > 0$ if and only if $d \in [d^{(n)}, d^{(n+1)})$,
(A2) for any $n \in \mathbb{N}_+$, if $d, d' \geq d^{(n)}$, then $d + d' \geq d^{(n+1)}$ (*i.e.* $2d^{(n)} \geq d^{(n+1)}$),
(A3) $\lim_{n\to\infty} d^{(n)} = \infty$.

For example one could take $d^{(n)} = n$ or $d^{(n)} = a^{n-1}$ with $a \in (1,2]$.

The HCP is then given by a sequence of one-epoch coalescence processes, suitably linked. More precisely, at the beginning of the first epoch one starts with a SPP with support on $\mathcal{N}(d^{(1)}) = \mathcal{N}(1)$. Then the stochastic evolution of the HCP is described by the sequence of random paths $\{\xi^{(n)}(\cdot)\}_{n\geq 1}$, where each $\xi^{(n)}$ is the random trajectory of the OCP with rates $\lambda_\ell^{(n)}$, $\lambda_r^{(n)}$, $\lambda_a^{(n)}$, active domain lengths $d_{\min}^{(n)} = d^{(n)}$, $d_{\max}^{(n)} = d^{(n+1)}$ and initial condition $\xi^{(n)}(0) = \xi^{(n-1)}(\infty)$, $n \geq 2$. Informally we refer to $\xi^{(n)}$ as describing the evolution in the n^{th}-epoch. Note that, by Lemma 2.4, one can prove recursively that at the end of the n^{th}–epoch the random configuration $\xi^{(n)}(\infty)$ belongs to $\mathcal{N}(d^{(n+1)})$, hence it is an admissible starting configuration for the OCP associated to the $(n+1)^{th}$–epoch.

Lemma 2.5 gives us information on the evolution and its asymptotics inside each epoch when the initial condition is a SPP of the renewal type. If *e.g.* the initial distribution $\mathcal{Q}$ for the first epoch is $\text{Ren}(\nu,\mu)$, where μ has support on $[d^{(1)}, \infty) = [1,\infty)$, we can use Lemma 2.5 together with the link $\xi^{(n+1)}(0) = \xi^{(n)}(\infty)$ between two consecutive epochs to recursively define the measures $\mu^{(n)}, \nu^{(n)}$ by

$$\begin{aligned} \mu^{(n+1)} &:= \mu_\infty^{(n)}, \quad \mu^{(1)} := \mu\,, \\ \nu^{(n+1)} &:= \nu_\infty^{(n)}, \quad \nu^{(1)} := \nu. \end{aligned} \tag{21}$$

With this position it is then natural to ask if, in some suitable sense, the measures $\mu^{(n)}, \nu^{(n)}$ have a well defined limiting behaviour as $n \to \infty$. The affirmative answer is contained in the following theorem, which is the core of the paper, for some specific choice of transition rates. Before stating it we recall a useful result on the Laplace transform of probability measures on $[1,\infty)$.

Lemma 2.11 ([FMRT0]). *Let μ be a probability measure on $[1,\infty)$ and let $g(s)$ be its Laplace transform, i.e. $g(s) = \int e^{-sx}\mu(dx)\,, s \in \mathbb{R}_+$.*

i) If

$$\lim_{s\downarrow 0} -\frac{s g'(s)}{1-g(s)} = c_0, \tag{22}$$

then necessarily $0 \leq c_0 \leq 1$.

ii) The existence of the limit (22) *holds if:*
a) μ *has finite mean and then* $c_0 = 1$ *or*
b) for some $\alpha \in (0,1)$ μ *belongs to the domain of attraction of an* α*–stable law or, more generally,* $\mu\big((x,\infty)\big) = x^{-\alpha}L(x)$ *where* $L(x)$ *is a slowly varying*[1] *function at* $+\infty$, $\alpha \in [0,1]$, *and in this case* $c_0 = \alpha$.

The reader may find the proof in [FMRT0, Appendix A] together with an example for which the limit (22) does not exist.

Theorem 2.12. *Let* ν, μ *be probability measures on* $\mathbb{R}$ *and* $[1,\infty)$ *respectively. Suppose that*

- *the law* $\mathcal{Q}$ *of* $\xi^{(1)}(0)$ *is either* $\mathcal{Q} = Ren(\nu,\mu)$ *or* $\mathcal{Q} = Ren(\mu)$ *or* $\mathcal{Q} = Ren_{\mathbb{Z}}(\mu)$,
- *it holds (i)* $\lambda_a^{(n)} \equiv 0$ *for all* $n \geq 1$, *or (ii)* $\lambda_\ell^{(n)} + \lambda_r^{(n)} \equiv \gamma\lambda_a^{(n)}$ *for all* $n \geq 1$ *and for some* $\gamma \geq 0$ *independent from* n,
- *the Laplace transform* $g(s)$ *of* μ *satisfies* (22).

For any $n \geq 1$ *let* $X^{(n)}$ *be a random variable with law* $\mu^{(n)}$ *defined in* (21) *so that* $g(s) := \mathbb{E}\big[e^{-sX^{(1)}}\big]$.

Then the following holds:

- *If* $c_0 = 0$, *then the rescaled variable* $Z^{(n)} = X^{(n)}/d^{(n)}$ *weakly converges to the random variable* $Z_0^{(\infty)} = \infty$.
- *If* $c_0 \in (0,1]$, *then the rescaled variable* $Z^{(n)} = X^{(n)}/d^{(n)}$ *weakly converges to the random variable* $Z_\kappa^{(\infty)}$ *with values in* $[1,\infty)$, *whose Laplace transform is given by*

$$g_\kappa^{(\infty)}(s) = \mathcal{R}\left(\kappa\int_1^\infty \frac{e^{-sx}}{x}dx\right), \qquad s > 0, \tag{23}$$

where

$$\begin{cases} \kappa := c_0 & \text{and} \quad \mathcal{R}(x) := 1 - e^{-x} & \text{in case (i)}, \\ \kappa := \frac{\gamma+1}{\gamma+2}c_0 & \text{and} \quad \mathcal{R}(x) := \dfrac{\exp\left\{\frac{\gamma+2}{\gamma+1}x\right\}-1}{\exp\left\{\frac{\gamma+2}{\gamma+1}x\right\}+\frac{1}{\gamma+1}} & \text{in case (ii)}. \end{cases} \tag{24}$$

The proof of Theorem 2.12 is given in Section 6. Case (i) has already been proved in [FMRT0] with a more combinatorial method, not suited for extensions.

Remark 2.13. *In the above result the only reminiscence of the initial distribution is through the constant* c_0 *which is "universal" for a large class of initial interval laws* μ *(see Lemma 2.11). In particular, starting with a stationary or* $\mathbb{Z}$*–stationary renewal SPP (which necessarily corresponds to a law* μ *with finite mean), the weak limit of* $Z^{(n)}$ *always exists and is universal* ($c_0 = 1$)*, depending on the rates only through the fulfilment of case (i) or case (ii), and not depending on the sequence* $\{d^{(n)}\}_{n\geq 1}$ *which defines the active intervals.*

We also underline that our results cover a slightly more general class of HCP. Indeed, following exactly the same lines of our proofs, we can also treat more general triple merging allowing e.g. an active domain to incorporate either its two neighbours to the left and/or its two neighbours to the right and/or its left and right neighbours (this last case is the only one considered in this paper). For this more general class

[1] A function L is said to be slowly varying at infinity, if for all $c > 0$, $\lim_{x\to\infty} L(cx)/L(x) = 1$.

of HCP both the above asymptotic result as well as the one in Theorem 2.15 are unchanged (instead the single epoch evolution expressed by the differential equation (29) *has to be properly changed by adding to λ_a the rates of these new triple mergence events).*

Remark 2.14. *The asymptotic Laplace distribution $g_\kappa^{(\infty)}$ can be written also as*

$$g_\kappa^{(\infty)}(s) = \mathcal{R}\left(\kappa \int_s^\infty \frac{e^{-x}}{x} dx\right) = \mathcal{R}\left(\kappa\, Ei(s)\right)$$

where $Ei(\cdot)$ denotes the exponential integral function[2]*. This is indeed the form appearing in* [DBG] *and* [SE]*. Moreover, in case (ii) with $\gamma = 0$ in the above theorem (as in* [DBG]*), one simply has $\kappa = c_0/2$ and $g_{c_0/2}^{(\infty)} = \tanh\left(\frac{c_0}{2} Ei(s)\right)$.*

Next we concentrate on the asymptotic behaviour of the first point law when starting with a right renewal SPP.

Theorem 2.15. *Let ν, μ be probability measures on $\mathbb{R}$ and $[1, \infty)$ respectively. Suppose that*

- *the law $\mathcal{Q}$ of $\xi^{(1)}(0)$ is $Ren(\nu, \mu)$,*
- *it holds (i) $\lambda_a^{(n)} \equiv 0$ and $\lambda_r^{(n)} \equiv \gamma \lambda_\ell^{(n)}$ for all $n \geq 1$ and for some $\gamma \geq 0$ independent from n, or (ii) $\lambda_\ell^{(n)} \equiv 0$ and $\lambda_r^{(n)} \equiv 0$ for all $n \geq 1$.*
- *the Laplace transform $g(s)$ of μ satisfies* (22)*.*

For any $n \geq 1$ let $X_0^{(n)}$ be the position of the first point of the HCP at the beginning of the n–th epoch and let $Y^{(n)}$ be the rescaled random variable $Y^{(n)} := X_0^{(n)}/d^{(n)}$.

Then the following holds:

- [FMRT0] *In case (i), as $n \to \infty$, $Y^{(n)}$ weakly converges to the positive random variable $Y_{c_0}^{(\infty)}$ with Laplace transform given by*

$$\mathbb{E}\big(e^{-sY_{c_0}^{(\infty)}}\big) = \exp\left\{-\frac{c_0}{1+\gamma}\int_{(0,1)} \frac{1-e^{-sy}}{y} dy\right\}, \qquad s \in \mathbb{R}_+. \tag{25}$$

- *In case (ii), supposing that $\int z\mu(dz) < \infty$ and that*

$$\lim_{z\to\infty} \frac{1}{z}\int_{[1,z]} x^2 \mu(dx) = 0, \tag{26}$$

as $n \to \infty$ the variable $Y^{(n)}$ weakly converges to the random variable $Y^{(\infty)}$ with values in $(0, \infty)$ and Laplace transform given by

$$\mathbb{E}\big(e^{-sY^{(\infty)}}\big) = \frac{e^{-\bar\gamma/2}}{2}\sqrt{\frac{1-\tanh^2(\mathrm{Ei}(s)/2)}{s}}, \qquad s \in \mathbb{R}_+, \tag{27}$$

where we let $\bar\gamma = -\int_0^\infty e^{-t}(\log t)dt \simeq 0,577$ be the Euler-Mascheroni constant. Condition (26) *is satisfied if $\int x^{1+\varepsilon}\mu(dx) < \infty$ for some $\varepsilon > 0$.*

The proof is given in Section 7. Case (i) in the above theorem has been stated only for completeness. It has already been obtained in [FMRT0] (see Theorem 2.24 there) . Finally, we point out that, due to Lemma 2.11, under condition (26) it must be $c_0 = 1$ in the limit (22).

[2]Note that the function that we denote by $\mathrm{Ei}(s)$ (following the notation of [SE] and our paper [FMRT0]) is instead more frequently denoted in mathematics literature by $\mathrm{E}_1(s)$.

Remark 2.16. *Extensions of the results presented in this section to OCP's and HCP's starting from an exchangeable SPP can be easily achieved following the arguments reported in* [FMRT0, Appendix D].

2.4. **Examples of HCP's.** We conclude this section by discussing some HCP's coming from the physics literature.

2.4.1. *The HCP associated to the East model at low temperature* [SE, FMRT1]*.* An interesting and highly non trivial example of HCP has been devised in physics literature [SE] to model the high density (or low temperature) non-equilibrium dynamics of the *East model* when a deep quench from a normal density state is performed. The East model [SE, EJ] is a well known example of kinetically constrained stochastic particle system with site exclusion which evolves according to a Glauber dynamics submitted to the following constraint: the 0/1 occupancy variable at a given site $x \in \mathbb{Z}$ can change only if the site $x+1$ is empty (*i.e.* the corresponding occupation variable equals zero). The change of the occupation variable, when allowed by this constraint, occurs at rate q (respectively $1-q$) if it corresponds to a change towards an empty (respectively occupied) site. Note that each configuration can also be represented by a sequence of domains on $\mathbb{Z}$, where a domain represents a maximal sequence of consecutive occupied sites delimited by two empty sites. If the equilibrium vacancy density is very low (*i.e.* in the limit $q \to 0$) and the initial distribution has a normal density (*e.g.* $q = 1/2$) most of the non-equilibrium evolution will try to remove the excess of vacancies of the initial state and will thus be dominated by the coalescence of domains. In this setting, under a proper rescaling [FMRT1], the East process can be well described by an HCP with the following parameters: $d^{(1)} = 1$, $d^{(n)} = 2^{n-2} + 1$ for $n \geq 2$, $\lambda_r^{(n)}(d) = \lambda_a^{(n)}(d) = 0$ for any value of the domain length d, thus $\lambda^{(n)} = \lambda_\ell^{(n)}$ where $\lambda_\ell^{(n)}$ is a function expressed via a proper large deviation probability (see [FMRT1] for the precise form of this function). We provide here only a very short explanation to justify the above choices of the parameters and refer the reader to [SE] for an heuristic explanation of the connection of this HCP with East and to Section 3 of [FMRT1] for a rigorous description. The choice $\lambda_a^{(n)} = 0$ is due to the fact that the relevant event for East corresponds to the disappearance of one zero at a time, namely to the coalescence of two domains (triple domain merging is not allowed). The asymmetry between the right and left coalescence is due to the orientated character of the East constraints which implies that only the left domains can be incorporated. Finally, the apparently weird choice of the active ranges $d^{(n)}$ is due to the fact that in order to remove the vacancy sitting at the left border of a domain of size $\ell \in [2^{n-1}+1, 2^n]$ one needs to create at least n additional vacancies inside the domain (again, see [SE, FMRT1] for details of the combinatorial argument leading to this result). Thus energy barrier considerations imply that this event requires a typical time of order $1/q^n$ which in turn means that in the regime $q \to 0$ domains of sizes ℓ, ℓ' with $\ell \in [2^{n-1}+1, 2^n]$, $\ell' \in [2^{m-1}+1, 2^m]$ and $n \neq m$ are active (namely their left border can disappear) on very well separated time scales.

2.4.2. *The paste–all model* [DGY2]*.* Another interesting HCP has been "introduced" in [DGY2] and named *Paste-all-model.* The model was intended to describe breath figures, namely the patterns formed by growing and coalescing droplets when vapour condenses on a non wetting surface. A common feature of breath figure experiments is the occurrence of a scale-invariant regime with a stable distribution of the drop

sizes. In [DGY2] several simplified one dimensional models were proposed to understand this phenomenon, including the HCP named Paste-all-model. In this case all the domains are subintervals of the integer lattice, a single length is active in each epoch and domains merge with their left/right neighbour with rate one, namely $d^{(n)} = n$, $\lambda_\ell^{(n)}(n) = \lambda_r^{(n)}(n) = 1$ and $\lambda_a^{(n)}(n) = 0$ (drops can coalesce either with their right or left neighbour and the smaller droplets are the first that disappear).

2.4.3. *The HCP associated to the 1d Ising model* [BDG]. Finally, we recall the HCP which has been "introduced" in [BDG] to model the zero temperature Glauber dynamics of the one dimensional Ising model evolved from a random initial condition. In this case the domains correspond to the ordered spin regions, namely the maximal sequence of consecutive sites with the same value of the spin, either up or down. At late stages of the dynamics a scale-invariant morphology develops: the structure at different times is statistically similar apart from an overall change of scale, *i.e.* the system is described by a single, time-dependent length scale. Instead of considering the stochastic Glauber dynamics the authors of [BDG] start from the well known simpler deterministic model which is expected to mimics this dynamics, namely the time-dependent Ginzburg-Landau equation for a scalar field in $d = 1$, $\partial_t \phi = \partial_x^2 \phi - dV/d\phi$ with $V(\phi)$ a symmetric double well potential with minima at $\phi = \pm 1$ corresponding to the up and down phases for the Ising model. If the model starts with a ϕ profile corresponding to a random initial condition for the Ising model, then it evolves rapidly to a phase of subsequent regions were ϕ is close to ± 1 (corresponding to the ordered domains) and the dynamics is dominated by the events that bring together and annihilate the closest pair of domain walls. This in turn corresponds to the fact that the smaller domains merge with the two neighbouring domains. Consequently, the HCP which has been introduced in [BDG] to mimic this domain dynamics has parameters: $d^{(n)} := n$ (only the smallest length is active at each epoch), $\lambda_\ell^{(n)}(n) = \lambda_r^{(n)}(n) = 0$ and $\lambda_a^{(n)}(n) = 1$ (only triple merging occurs).

3. Metric structure of $\mathcal{N}(d_{\min})$

Let us write $\mathcal{M}$ for the space of Radon measures on $\mathbb{R}$, *i.e.* locally finite Borel non-negative measures. We consider this space endowed of the vague topology, such that $\mu_n \to \mu$ in $\mathcal{M}$ if and only if $\mu_n(f) \to \mu(f)$ for all continuous functions f on $\mathbb{R}$ with compact support (shortly, $f \in C_0$). Then $\mathcal{M}$ can be metrizied by a suitable metric m making it a Polish space (see [DV, Sec. A2.6] and observe that, since the Euclidean space $\mathbb{R}$ is Polish and locally compact, the vague topology coincides with the $\hat{w}$–topology as discussed before [DV, Cor. A2.6.V]). We recall the definition of m since useful below:

$$m(\mu, \nu) := \int_0^\infty e^{-r} \frac{d_r\big(\mu^{(r)}, \nu^{(r)}\big)}{1 + d_r\big(\mu^{(r)}, \nu^{(r)}\big)} dr, \qquad \mu, \nu \in \mathcal{M},$$

where $\mu^{(r)}$, $\nu^{(r)}$ denote the restriction to $(-r, r)$ of μ, ν, while d_r stands for the Prohorov distance for measures on $(-r, r)$ (see [DV, Sec. A2.5]).

The space $\mathcal{N}$ introduced in Section 2.1 can be thought of as a subspace of $\mathcal{M}$, identifying the set $\xi \in \mathcal{N}$ with the measure $\sum_{x \in \xi} \delta_x$. Then one gets that the σ–algebra of its Borel subsets coincides with the σ–algebra of measurable subsets introduced in Section 2.1 (see [DV, Ch. 7], in particular Prop.7.1.III and Cor. 7.1.VI there).

Therefore, the same property holds for $\mathcal{N}(d_{\min})$ (*i.e.* Borel subsets and measurable subsets coincide).

Lemma 3.1. *The following holds:*

(*i*) *$\xi_n \to \xi$ in $\mathcal{N}(d_{\min})$ if and only if $|\xi_n \cap [a,b]| \to |\xi \cap [a,b]|$ for each interval $[a,b]$ such that $\xi \cap \{a,b\} = \emptyset$. The same criterion holds replacing closed intervals $[a,b]$ by open intervals (a,b).*

(*ii*) *Suppose that $\xi_n \to \xi$ in $\mathcal{N}(d_{\min})$. Fix $a < b$ with $\xi \cap \{a,b\} = \emptyset$. Then $\xi_n \cap (a,b) \to \xi \cap (a,b)$ in $\mathcal{N}(d_{\min})$. Moreover, for n large enough $\xi_n \cap (a,b)$ has the same cardinality of $\xi \cap (a,b)$ and $\lim_{n\to\infty} x_i^{(n)} = x_i$ for $1 \le i \le k$, where $\xi_n \cap (a,b) = \{x_1^{(n)} < x_2^{(n)} < \cdots < x_k^{(n)}\}$ and $\xi \cap (a,b) = \{x_1 < x_2 < \cdots < x_k\}$.*

(*iii*) *The space $\mathcal{N}(d_{\min})$ is a closed subset of $\mathcal{M}$. In particular, it is a Polish space endowed of the metric m.*

(*iv*) *The space $\mathcal{N}(d_{\min})$ is compact.*

Proof. Part (*i*) with closed intervals follows from [DV, Prop. A2.6.II] (see also [FKAS, Th.1.1.16] with $P_n := \delta_{\xi_n}$ and $P := \delta_\xi$). The same criterion with open interval is a simple derivation from the one with closed interval.

Let us consider Part (*ii*). Applying the criterion in Part (*i*) it is trivial to check that $\xi_n \cap (a,b) \to \xi \cap (a,b)$. Take now $\varepsilon > 0$ small enough that all the intervals $J_i = [x_i - \varepsilon, x_i + \varepsilon]$, $1 \le i \le k$, are disjoint and intersect ξ only at x_i. Then, by item (*i*) for n large ξ_n has exactly one point in each J_i. Similarly, ξ_n has exactly k points in (a,b) for n large. By the arbitrariness of ϵ we can conclude.

To prove Part (*iii*) call $\bar{\mathcal{N}}$ the family of counting measures in $\mathbb{R}$, *i.e.* $\xi \in \bar{\mathcal{N}}$ if and only if $\xi = \sum_i k_i \delta_{x_i}$ with $k_i \in \mathbb{N}_+$ and $\{x_i\}$ being a locally finite countable subset of $\mathbb{R}$. By [DV, Prop.7.1.III], $\bar{\mathcal{N}}$ is a closed subset of $\mathcal{M}$. Hence, if $\xi_n \in \mathcal{N}(d_{\min})$ and $\xi_n \to \xi$ with ξ in $\mathcal{M}$, then $\xi \in \bar{\mathcal{N}}$. We only need to show that $\xi \in \mathcal{N}(d_{\min})$. Suppose by contradiction that $\xi(\{x\}) \ge 2$ for some $x \in \mathbb{R}$. Take $I = [x-\epsilon, x+\epsilon]$ such that $\xi(\{x-\epsilon, x+\epsilon\}) = 0$ and $2\varepsilon < d_{\min}$ (the existence of ϵ is guaranteed by the fact that $\xi \in \bar{\mathcal{N}}$). By Part (*i*) it must be $\xi_n(I) \ge 2$ for n large enough, in contradiction with the fact that ξ_n can have at most one point in I.

Due to Part (*iii*), Part (*iv*) is a simple consequence of the compactness criterion given in [DV, Cor. A2.6.V]. □

Recall that $\mathbb{B}$ denotes the Banach space of all continuous functions $f : \mathcal{N}(d_{\min}) \to \mathbb{R}$ endowed with the uniform norm $\|\cdot\|$ and that $\mathbb{B}_{\text{loc}}$ denotes the set of local functions $f \in \mathbb{B}$.

Lemma 3.2. *The set $\mathbb{B}_{\text{loc}}$ is dense in $\mathbb{B}$. In particular, given $f \in \mathbb{B}$ and defining $f_N(\xi) := \int_N^{N+1} f(\xi \cap (-r,r))dr$, it holds $f_N \in \mathbb{B}_{\text{loc}}$ and $f_N \to f$ in $\mathbb{B}$.*

Note that the map $\mathbb{R}_+ \ni r \mapsto f(\xi \cap (-r,r)) \in \mathbb{R}$ is stepwise, with a finite number of jumps in any finite interval. Hence, the above function f_N is well defined.

Proof. Take $f \in \mathbb{B}$. Since $\mathcal{N}(d_{\min})$ is compact, f is uniformly continuous. Hence, given $\epsilon > 0$, there exists $\delta_0 > 0$ such that $m(\xi,\xi') < \delta_0$ implies $|f(\xi) - f(\xi')| < \epsilon$. Take $N_0 \in \mathbb{N}$ large enough that $e^{-N_0} \le \delta_0$. By the definition of m we have $m(\xi, \xi \cap (-N,N)) \le \int_N^\infty e^{-a} da \le \delta_0$ for any $N \ge N_0$. This implies that $|f(\xi) - f(\xi \cap (-r,r))| \le \epsilon$ for all $r \ge N_0$ and therefore $\|f - f_N\| \le \epsilon$. Trivially f_N is a local function, it remains to prove that f_N is continuous. To this aim, fix $\xi \in \mathcal{N}(d_{\min})$. Then the set $R = \{r \in [N, N+1] : \xi \cap \{-r, r\} \ne \emptyset\}$ is finite. In particular, by

Lemma 3.1 (*ii*), if $\xi_n \to \xi$ then $\xi_n \cap (-r,r) \to \xi \cap (-r,r)$ for all $r \in [N, N+1] \setminus R$. Since f is continuous, we get that

$$f(\xi_n \cap (-r,r)) \to f(\xi \cap (-r,r)) \qquad \forall r \in [N, N+1] \setminus R.$$

We conclude applying now the dominated convergence theorem. □

4. OCP process: proof of Theorem 2.6 and Theorem 2.8

In this section we prove Theorem 2.6 and Theorem 2.8 applying our analysis of the Markov generator of the OCP (recall Corollary 2.10).

4.1. **Differential equation for μ_t and proof of Theorem 2.6.** As application of Corollary 2.10 we can prove the following result:

Proposition 4.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that*

$$\sum_{k=0}^{\infty} \sup_{x \geq k} |f(x)| < \infty. \tag{28}$$

Let μ be a probability measure on $[d_{\min}, \infty)$ and μ_t be as in Lemma 2.5 with the choice $Q = Ren(\mu)$. Then, the function $[0,\infty) \ni t \mapsto \mu_t(f) \in \mathbb{R}$ is differentiable and

$$\begin{aligned}\frac{d}{dt}\mu_t(f) = &-\int \mu_t(dx)\lambda(x) f(x) + \int \mu_t(dx) \int \mu_t(dy)\big(\lambda_r(x) + \lambda_\ell(y)\big) f(x+y) \\ &+ \int \mu_t(dx) \int \mu_t(dy) \int \mu_t(dz) \lambda_a(y) f(x+y+z).\end{aligned} \tag{29}$$

Proof. Set $\mathcal{Q} = \text{Ren}(\delta_0, \mu)$. Note that $\mathbb{P}_{\mathcal{Q}}$–a.s. $\xi(t)$ belongs to the set $\mathcal{N}_*$ of configurations $\xi \in \mathcal{N}(d_{\min})$ such that $\xi \subset [0,\infty)$, $\xi \cap (0, d_{\min}/2] = \emptyset$ and ξ is given by an increasing sequence of points diverging to ∞. Points in $\xi \in \mathcal{N}_*$ are labeled as $x_0(\xi), x_1(\xi), x_2(\xi), \dots$ in increasing order. Then, by Lemma 2.5, μ_t equals the law of $x_1(\xi(t))$ under $\mathbb{P}_{\mathcal{Q}}(\cdot | 0 \in \xi(t))$. Hence we can write $\mu_t(f) = N_t / D_t$ where

$$N_t = \mathbb{E}_{\mathcal{Q}}\left[f\big(x_1(\xi(t))\big); 0 \in \xi(t)\right], \qquad D_t = \mathbb{P}_{\mathcal{Q}}(0 \in \xi(t)).$$

Let $\rho : \mathbb{R} \to [0,1]$ be a continuous function such that $\rho(x) = 0$ for $x \notin (-\frac{d_{\min}}{2}, \frac{d_{\min}}{2})$ and $\rho(0) = 1$. By definition of vague convergence (see Section 3), the function $\Phi : \mathcal{N}(d_{\min}) \mapsto \mathbb{R}$ defined as $\Phi(\xi) := \sum_{x \in \xi} \rho(x)$ is a continuous map. Since local it belongs to $\mathbb{B}_{\text{loc}}$ and moreover it satisfies $\Phi(\xi) = \mathbb{1}_{0 \in \xi}$ for all $\xi \in \mathcal{N}_*$. In Lemma 4.2 below we exhibit a function $\Psi \in \mathbb{D}$ that satisfies $\Psi(\xi) = f\big(x_1(\xi)\big)\mathbb{1}_{0 \in \xi}$ for all $\xi \in \mathcal{N}_*$. Hence, we can write

$$N_t = \mathbb{E}_{\mathcal{Q}}\left[\Psi(\xi(t))\right], \qquad D_t = \mathbb{E}_{\mathcal{Q}}\left[\Phi(\xi(t))\right].$$

By standard properties of Markov generators, we conclude that the maps N_t, D_t are differentiable and that

$$N'_t = \mathbb{E}_{\mathcal{Q}}\left[\mathcal{L}\Psi(\xi(t))\right], \qquad D'_t = \mathbb{E}_{\mathcal{Q}}\left[\mathcal{L}\Phi(\xi(t))\right].$$

Since $\Psi, \Phi \in \mathbb{D}$, we can use Equation (3) to compute $\mathcal{L}\Psi$ and $\mathcal{L}\Phi$. We need their value only on $\mathcal{N}_*$. Suppose that $\zeta, \xi \in \mathcal{N}_*$ are such that $\zeta \subset \xi$ and $0 \in \xi$. Writing x_i and d_i instead of $x_i(\zeta)$ and $d_i(\zeta) = x_i(\zeta) - x_{i-1}(\zeta)$, we get

$$\begin{cases}\mathcal{L}\Psi(\zeta) = \mathbb{1}(0 \in \zeta) G(\zeta), \\ \mathcal{L}\Phi(\zeta) = \mathbb{1}(0 \in \zeta) H(\zeta)\end{cases}$$

where $H(\zeta) := -\lambda_\ell(d_1) - \lambda_a(d_1)$ and

$$G(\zeta) := -\big[\lambda_\ell(d_1) + \lambda_a(d_1)\big] f(x_1) + \big[\lambda_r(d_1) + \lambda_\ell(d_2)\big]\big[f(x_2) - f(x_1)\big] \\ + \lambda_a(d_2)\big[f(x_3) - f(x_1)\big].$$

Since N_t and D_t are derivable, we get that N_t/D_t is derivable and that

$$\frac{d}{dt}\mu_t(f) = \frac{d}{dt}\frac{N_t}{D_t} = \frac{N_t'}{D_t} - \frac{N_t}{D_t}\frac{D_t'}{D_t}.$$

Writing $F(\xi) = f(x_1(\xi))$, the above identities imply that

$$\frac{d}{dt}\mu_t(f) = \mathbb{E}_{\mathcal{Q}}\left(G(\xi(t))\,|\,0 \in \xi(t)\right) - \mathbb{E}_{\mathcal{Q}}\left(F(\xi(t))\,|\,0 \in \xi(t)\right)\mathbb{E}_{\mathcal{Q}}\left(H(\xi(t))\,|\,0 \in \xi(t)\right). \tag{30}$$

By Lemma 2.5 (iv), we can write (brackets should help to follow the computations)

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left(G(\xi(t))\,|\,0 \in \xi(t)\right) &= -\left\{\mu_t(\lambda_\ell f) + \mu_t(\lambda_a f)\right\} \\ &+ \left\{\int \mu_t(dx)\int \mu_t(dy)\big[\lambda_r(x) + \lambda_\ell(y)\big]f(x+y) - \mu_t(\lambda_r f) - \mu_t(\lambda_\ell)\mu_t(f)\right\} \\ &+ \left\{\int \mu_t(dx)\int \mu_t(dy)\int \mu_t(dz)\lambda_a(y)f(x+y+z) - \mu_t(\lambda_a)\mu_t(f)\right\}\end{aligned} \tag{31}$$

and

$$\mathbb{E}_{\mathcal{Q}}\left(F(\xi(t))\,|\,0 \in \xi(t)\right)\mathbb{E}_{\mathcal{Q}}\left(H(\xi(t))\,|\,0 \in \xi(t)\right) = -\mu_t(f)\mu_t(\lambda_\ell) - \mu_t(f)\mu_t(\lambda_a). \tag{32}$$

Combining the above identities (30), (31) and (32) we get the thesis. □

In the proof of Proposition 4.1 above, we used the following technical lemma.

Lemma 4.2. *Let f be a real continuous function on $[0,\infty)$ satisfying (28) and extend it to a continuous function on $\mathbb{R}$ constant on $(-\infty, 0]$. Given $s \in \mathbb{R}$ define*

$$f_s(\xi) = \begin{cases} 0 & \text{if } \big|\xi \cap (s,\infty)\big| \le 1 \\ f\left(z\big(\xi \cap (s,\infty)\big)\right) & \text{otherwise}\end{cases}$$

where $z\big(\xi \cap (s,\infty)\big)$ denotes the second point from the left of $\xi \cap (s,\infty)$. Then the function

$$F : \mathcal{N}(d_{\min}) \ni \xi \mapsto \frac{1}{d_{\min}}\int_{-d_{\min}}^{0} f_s(\xi)ds \in \mathbb{R}$$

belongs to $\mathbb{B}$. Moreover, the function $\Psi(\xi) = \Phi(\xi)F(\xi)$ belongs to $\mathbb{D}$ and $\Psi(\xi) = f\big(x_1(\xi)\big)\mathbb{1}_{0\in\xi}$ for all $\xi \in \mathcal{N}_*$ (for the definition of Φ and $\mathcal{N}_*$ see the the proof of Proposition 4.1).

The integrand in the definition of F is a stepwise function with a finite number of jumps, hence it is integrable.

Proof. Let us prove the continuity of F. Take $\xi_n \to \xi$ in $\mathcal{N}(d_{\min})$ and set $R := \{s \in (-d_{\min}, 0) : s \notin \xi\}$. We claim that, fixed $s \in R$, it holds $f_s(\xi_n) \to f(\xi)$. Let us first suppose that $|\xi \cap (s,\infty)| \ge 2$. Let $a < b$ be the first two points of $\xi \cap (s,\infty)$ and take c larger than b such that ξ has no point in $(b, c]$. Then by Lemma 3.1 (ii) $\xi_n \cap (s, c)$ has exactly two points $a^{(n)} < b^{(n)}$ eventually in n, moreover $a^{(n)} \to a$ and $b^{(n)} \to b$. By the continuity of f, we have

$$f_s(\xi_n) = f(b^{(n)}) \to f(b) = f_s(\xi).$$

Let us now suppose that $|\xi \cap (s,\infty)| \leq 1$. Suppose first that $\xi \cap (s,\infty)$ has only one point, denoted by x_*. Given $\varepsilon > 0$ fix $L > x_*$ such that $L \notin \xi$ and $|f(x)| \leq \varepsilon$ for $x \geq L$ (L exists due to (28)). By Lemma 3.1 (i) for n large ξ_n has exactly one point in (s, L). This assures that $|f_s(\xi_n)| \leq \varepsilon$ for n large and therefore that $\lim_{n\to\infty} f_s(\xi_n) = 0 = f_s(\xi)$. A similar argument can be applied when ξ has no point in (s,∞). This concludes the proof of our claim.

Combining our claim with the dominated convergence theorem and with the fact that R is a finite set, we get that $F(\xi_n) \to F(\xi)$, thus proving the continuity of F.

If $\xi \in \mathcal{N}_*$ it is simple to check that $\Psi(\xi) = f\big(x_1(\xi)\big)\mathbb{1}_{0\in\xi}$. It remains to prove that $|||\Psi||| < \infty$. Suppose that $k \in \mathbb{Z}$ and $\nabla_k \Psi(\xi) \neq 0$. Then $k \geq -1$ and ξ has at least two points in $(-d_{\min}, \infty)$, the first or the second one (from the left) must lie in I_k. In particular, it must be $|\nabla_k \Psi(\xi)| \leq 2 \sup_{x \geq k d_{\min}} |f(x)|$. Take now $k < k' \leq k + \lceil d_{\max}/d_{\min} \rceil$ in $\mathbb{Z}$. Suppose that $\nabla_{(k,k')}\Psi(\xi) \neq 0$. If $k < -1$ then $\nabla_{(k,k')}\Psi(\xi) = \nabla_{k'}\Psi(\xi)$ which can be bounded as above. If $k, k' \geq -1$, then we conclude that ξ has at least two points in $(-d_{\min}, \infty)$, the first or the second one (from the left) must lie in $I_k \cup I_{k'}$. Hence, $|\nabla_{(k,k')}\Psi(\xi)| \leq 2 \sup_{x \geq k d_{\min}} |f(x)|$. The above bounds and condition (28) allow to conclude. □

We have now all the tools to prove Theorem 2.6.

Proof of Theorem 2.6. To prove (4) and (5) we can restrict to $s > 0$. Indeed, writing these differential equations as integral identities one can take the limit $s \downarrow 0$ and recover the case $s = 0$.

We can apply Proposition 4.1 to the function $f(x) = e^{-sx}$, $x \geq 0$, getting that $G_t(s) = \mu_t(f)$ is t–differentiable, with derivative given by (29).

We can write $H_t(s) = \mu_t(\tilde f)$ where $\tilde f(x) := e^{-sx}\mathbb{1}(x < d_{\max})$. Obviously $\tilde f$ is not suited to Proposition 4.1 since not continuous. If μ had support on a lattice, *e.g.* $\mathbb{N}$, trivially $\tilde f$ could be replaced by a nice function. In the general case we need more care. For $\varepsilon > 0$ small enough, we fix a continuous function f_ε on $[0,\infty)$ with values in $[0,1]$ such that $f_\varepsilon(x) = \tilde f(x)$ if $x \notin \big(d_{\max} - \varepsilon, d_{\max}\big)$. Applying Proposition 4.1 we get that the function $[0,\infty) \ni t \mapsto \mu_t(f_\varepsilon)$ is differentiable with derivative given by (29) (with f replaced by f_ε). Since μ_t has support in $[d_{\min}, \infty)$ and since $f_\varepsilon(x+y) = 0$, $f_\varepsilon(x+y+z) = 0$ if $x, y, z \geq d_{\min}$ (recall assumption (A2) in Section 2.2), from (29) we conclude that $\mu_t'(f_\varepsilon) = -\mu_t(\lambda f_\varepsilon)$.

Let $\alpha_t(\varepsilon) := \mu_t\big((d_{\max} - \varepsilon, d_{\max})\big)$. Trivially, $\lim_{\varepsilon\downarrow 0} \alpha_t(\varepsilon) = 0$ and $0 \leq \alpha_t(\varepsilon) \leq 1$. Since $|\mu_t(\tilde f) - \mu_t(f_\varepsilon)| \leq \alpha_t(\varepsilon)$ and

$$|\mu_t'(f_\varepsilon) + \mu_t(\lambda \tilde f)| = |\mu_t(\lambda f_\varepsilon) - \mu_t(\lambda \tilde f)| \leq \alpha_t(\varepsilon)\|\lambda\|_\infty,$$

applying the Dominated Convergence Theorem we get

$$H_t(s) = \mu_t(\tilde f) = \lim_{\varepsilon\downarrow 0} \mu_t(f_\varepsilon) = \lim_{\varepsilon\downarrow 0}\big[\mu_0(f_\varepsilon) - \int_0^t \mu_u(\lambda f_\varepsilon)du\big] = \mu_0(\tilde f) - \int_0^t \mu_u(\lambda \tilde f)du.$$

Since λ is a continuous function (extendable on $[0,\infty)$) and is zero on $[d_{\max}, \infty)$, we can apply Proposition 4.1 to the function $\lambda\tilde f$ concluding that the map $[0,\infty) \ni t \mapsto \mu_t(\lambda \tilde f)$ is differentiable and therefore continuous. This observation together with the above identity allows to conclude that $H_t(s)$ is t–differentiable and its derivative satisfies (4) (note that $\lambda \tilde f = \lambda f$ by assumption (A1) in Section 2.2). Knowing that $\partial_t G_t(s)$ is given by (29) and using (4) we get (5).

We observe that in case (i) it holds $\lambda_\ell + \lambda_r = \lambda$, while in case (ii) it holds $\lambda_\ell + \lambda_r = \lambda\gamma/(1+\gamma)$ and $\lambda_a = \lambda/(1+\gamma)$. These identities allow to derive from (4) and (5) that $\partial_t G_t(s) = \partial_t H_t(s)(1 - G_t(s))$ in case (i) and that $\partial_t G_t(s) = \partial_t H_t(s)\left(1 - \frac{G_t(s)(\gamma+G_t(s))}{1+\gamma}\right)$ in case (ii). The rest of the proof follows by the computations outlined in Remark 2.7 using (4), (5) and the fact that $\lim_{t\to\infty} G_t(s) = G_\infty(s)$, $\lim_{t\to\infty} H_t(s) = H_\infty(s) = 0$ (which is due to Lemma 2.4 (iii) and Lemma 2.5 (v)). We only point out that with the definition of $A_t(s), B_t(s)$ given in Remark 2.7 one gets $b_u(s)e^{A_u(s)+2B_u(s)} = \frac{1}{\gamma+2}\partial_u e^{-H_u\frac{\gamma+2}{\gamma+1}}$ in case (ii). □

4.2. **Differential equation for ν_t and proof of Theorem 2.8.** As in the case of the interval law μ_t, in order to prove Theorem 2.8, we need first to establish a differential equation for the expectation $\nu_t(f)$ for nice functions f.

Proposition 4.3. *Let $f : [0,\infty) \to \mathbb{R}$ be as in Proposition 4.1. Let μ be a probability measure on $[d_{\min},\infty)$ and ν_t be as in Lemma 2.5 with the choice $Q = Ren(\delta_0,\mu)$. Then the map $[0,\infty) \ni t \mapsto \nu_t(f) \in \mathbb{R}$ is differentiable and*

$$\frac{d}{dt}\nu_t(f) = -\int \nu_t(dx)\int \mu_t(dy)\left(\lambda_\ell(y)+\lambda_a(y)\right)f(x) + \int \nu_t(dx)\int \mu_t(dy)\lambda_\ell(y)f(x+y)$$
$$+ \int \nu_t(dx)\int \mu_t(dy)\int \mu_t(dz)\lambda_a(y)f(x+y+z). \tag{33}$$

Proof. We extend f as continuous function to all $\mathbb{R}$, constant on $(-\infty,0]$. Given $s \in \mathbb{R}$ define

$$f_s(\xi) = \begin{cases} 0 & \text{if } \xi\cap(s,\infty) = 0 \\ f\left(z\big(\xi\cap(s,\infty)\big)\right) & \text{otherwise} \end{cases}$$

where $z\big(\xi\cap(s,\infty)\big)$ denotes the first point from the left of $\xi\cap(s,\infty)$. Then the function

$$\Theta : \mathcal{N}(d_{\min}) \ni \xi \mapsto \frac{1}{d_{\min}}\int_{-d_{\min}}^0 f_s(\xi)ds \in \mathbb{R}$$

belongs to $\mathbb{D}$ and $\Theta(\xi) = f(x_0(\xi))$ if $\xi \in \mathcal{N}_*$. The proof is similar to the one of Lemma 4.2 and we omit the details.

Set $\mathcal{Q} = \mathrm{Ren}(\delta_0,\mu)$. Note that $\mathbb{P}_{\mathcal{Q}}$–a.s. $\xi(t)$ belongs to the set $\mathcal{N}_*$ of configurations $\xi \in \mathcal{N}(d_{\min})$ such that $\xi \subset [0,\infty)$, $\xi\cap(0,d_{\min}/2] = \emptyset$ and ξ is given by an increasing sequence of points diverging to ∞. Points in $\xi \in \mathcal{N}_*$ are labeled as $x_0(\xi), x_1(\xi), x_2(\xi), \dots$ in increasing order. Hence, we can write

$$\nu_t(f) = \mathbb{E}_{\mathcal{Q}}\big[f\big(x_0(\xi(t))\big)\big] = \mathbb{E}_{\mathcal{Q}}\big[\Theta(\xi(t))\big].$$

Using that $\Theta \in \mathbb{D}$ and therefore (3), one concludes that the map $t \to \nu_t(f)$ is differentiable and that

$$\begin{aligned}\frac{d}{dt}\nu_t(f) &= \mathbb{E}_{\mathcal{Q}}\big[\mathcal{L}\Theta(\xi(t))\big] \\ &= \mathbb{E}_{\mathcal{Q}}\left[\,\lambda_\ell(x_1-x_0)[f(x_1)-f(x_0)] + \lambda_a(x_1-x_0)[f(x_2)-f(x_0)]\,\right] \\ &= \int \nu_t(dx)\int \mu_t(dy)\lambda_\ell(y)[f(x+y)-f(x)] \\ &\quad+ \int \nu_t(dx)\int \mu_t(dy)\int \mu_t(dz)\lambda_a(y)[f(x+y+z)-f(x)]\end{aligned}$$

where we used, for simplicity of notation, $x_0 = x_0(\xi(t))$, $x_1 = x_1(\xi(t))$ and $x_2 = x_2(\xi(t))$ and the fact that x_0 has law ν_t, while $x_1 - x_0$ and $x_2 - x_1$ have law μ_t. □

Proof of Theorem 2.8. As in the proof of Theorem 2.6 we can take $s > 0$. Using Proposition 4.3 with the function $f : x \mapsto e^{-sx}$, we get that $t \mapsto L_t(s)$ is differentiable and that

$$\frac{d}{dt}L_t(s) = -\int \nu_t(dx)\int \mu_t(dy)\,(\lambda_\ell(y)+\lambda_a(y))\,e^{-sx} + \int \nu_t(dx)\int \mu_t(dy)\lambda_\ell(y)e^{-sx-sy}$$
$$+\int \nu_t(dx)\int \mu_t(dy)\int \mu_t(dz)\lambda_a(y)e^{-sx-sy-sz}.$$

The above equation corresponds to (13).

Consider the case $\lambda_a \equiv 0$. Then if $\lambda_\ell \equiv 0$, trivially $L_t = L_0$ for any $t \geq 0$. While for $\lambda_r \equiv \gamma\lambda_\ell$, one has $\lambda_\ell \equiv \frac{1}{1+\gamma}\lambda$ so that the differential equation (13) satisfied by L_t reads

$$\partial_t L_t(s) = \frac{L_t(s)}{1+\gamma}\left[-\int \mu_t(dy)\lambda(y) + \int \mu_t(dy)\lambda(y)e^{-sy}\right] = \frac{L_t(s)}{1+\gamma}\left(\partial_t H_0(s) - \partial_t H_t(s)\right)$$

where we used (4). Integrating and using that $\lim_{t\to\infty} H_t(s) = 0$ leads to (14) and (15).

Now consider the case $\lambda_\ell \equiv 0$, $\lambda_r \equiv 0$. Noticing that $\lambda_a \equiv \lambda$ and using (4), from (13) we obtain that $\partial_t \ln L_t(s) = \partial_t H_t(0) - G_t(s)\partial_t H_t(s)$. At this point we apply Point (ii) in Theorem 2.6 with $\gamma = 0$ getting for $s > 0$

$$\partial_t \ln L_t(s) = \partial_t H_t(0) - \frac{G_t(s)\partial_t G_t(s)}{1-G_t(s)^2} = \partial_t\left\{H_t(0) + \frac{1}{2}\ln(1-G_t(s)^2)\right\}.$$

This leads to (16), which implies (17) after taking the limit $t \to \infty$. □

5. Abstract generalization of the transformation introduced in [FMRT0]

We extend here a transformation developed in [FMRT0, Sec. 5] allowing to rephrase the non-linear identities on the Laplace transforms appearing in Theorem 2.6 and Theorem 2.8 into linear identities involving Radon measures. This transformation will be crucial in our analysis of the limiting behaviour of the HCP process (see Section 6 and 7).

Consider the OCP starting from a renewal SPP with interval law μ having support on $[d_{\min},\infty)$ (*i.e.* $\xi(0)$ has law $\mathrm{Ren}(\nu,\mu)$ or $\mathrm{Ren}(\mu)$ or $\mathrm{Ren}_{\mathbb{Z}}(\mu)$). We recall that μ_∞ denotes the interval law at the end of the epoch (see Lemma 2.5) and we call X_0, X_∞ some generic random variables with law μ, μ_∞, respectively. Then we define the rescaled random variables

$$Z_0 = X_0/d_{\min} \quad \text{and} \quad Z_\infty = X_\infty/d_{\max}$$

and we set, for $s \geq 0$,

$$g_0(s) = \mathbb{E}\big(e^{-sZ_0}\big)\,,\quad g_\infty(s) = \mathbb{E}\big(e^{-sZ_\infty}\big)\,,\quad h_0(s) = \mathbb{E}\big(e^{-sZ_0}; Z_0 < a\big)\,,\quad a := \frac{d_{\max}}{d_{\min}}.$$

By definition and because of Assumption (A2) and Lemma 2.4 (*iii*), we have that $Z_0 \geq 1$, $Z_\infty \geq 1$ and $a \in [1,2]$. In particular, $g_0(s), g_\infty(s) \in (0,1)$ for $s > 0$.

We observe that Equations (7) and (9) have the following common structure:

$$\mathcal{F}\big(g_\infty(as)\big) = \mathcal{F}\big(g_0(s)\big) - h_0(s)\,, \qquad s > 0,$$

where

$$\mathcal{F}(x) := \begin{cases} -\ln(1-x) & \text{for Equation (7)}, \\ \frac{\gamma+1}{\gamma+2}\ln\frac{1+x/(\gamma+1)}{1-x} & \text{for Equation (9)}\,. \end{cases} \tag{34}$$

With these examples in mind, we introduce the following definition.

Definition 5.1 (Hypothesis (H))**.** *We say that a real function $\mathcal{F}$ satisfies Hypothesis (H) if there exists $\varepsilon > 0$ such that $\mathcal{F}$ is defined on $(-\varepsilon, 1)$ and*

- *(H1) $\mathcal{F}$ is C^1,*
- *(H2) the derivative $\mathcal{F}'$ admits an analytic expansion on $(0,1)$ of the form $\mathcal{F}'(x) = \sum_{n=0}^{\infty} c_n x^n$ with $c_n \geq 0$ for all $n \geq 0$,*
- *(H3) $\mathcal{F}(0) = 0$, $\mathcal{F}$ is bijective from $(-\varepsilon, \varepsilon)$ to an open interval U containing 0 such that $\mathcal{R} := \mathcal{F}^{-1} : U \to (-\varepsilon, \varepsilon)$ is an analytic function and $\mathcal{R}'(0) = 1$ (i.e. $\mathcal{F}'(0) = 1$).*

By analytic expansion in (H3) we mean that $\mathcal{R}(x) = \sum_{k=1}^{\infty} r_k x^k$ for all $x \in U$, where the series in the r.h.s. if absolutely convergent.

One can easily verifies that both functions $\mathcal{F}$ defined in (34) satisfy Hypothesis (H) since for $|x| < 1$ we have the analytic expansions $-\ln(1-x) = x + x^2/2 + x^3/3 + \cdots$, while

$$\ln\frac{1+x/(\gamma+1)}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}\left[1 + (-1)^{n+1}(\gamma+1)^{-n}\right]. \tag{35}$$

Moreover, for $|x| < 1$, it holds

$$\mathcal{R}(x) = \begin{cases} 1 - e^{-x} & \text{if } \mathcal{F}(x) = -\ln(1-x), \\ \frac{\exp\left\{\frac{\gamma+2}{\gamma+1}x\right\} - 1}{\exp\left\{\frac{\gamma+2}{\gamma+1}x\right\} + \frac{1}{\gamma+1}} & \text{if } \mathcal{F}(x) = \frac{\gamma+1}{\gamma+2}\ln\frac{1+x/(\gamma+1)}{1-x}. \end{cases} \tag{36}$$

Finally, we introduce the following notation. Given an increasing function $\phi : [0,\infty) \to [0,\infty)$ and a Radon measure $\mathfrak{m}$ on $[0,\infty)$, we denote by $\mathfrak{m} \circ \phi$ the new Radon measure on $[0,\infty)$ defined by

$$\mathfrak{m} \circ \phi(A) = \mathfrak{m}(\phi(A)), \qquad A \subset \mathbb{R} \text{ Borel}.$$

Note that $\mathfrak{m} \circ \phi$ is indeed a measure, due to the injectivity of ϕ. Moreover, it holds

$$\int_0^\infty f(x)\mathfrak{m} \circ \phi(dx) = \int_{[\phi(0),\phi(\infty)]} f(\phi^{-1}(x))\mathfrak{m}(dx). \tag{37}$$

Above, and in what follows, we use the short notation $\int_0^\infty$ for $\int_{[0,\infty)}$.

Theorem 5.2. *Let $\mathcal{F}$ be a function satisfying Hypothesis (H). Then there exist unique Radon non-negative measures $t_0(dx)$ and $t_\infty(dx)$ on $[0,\infty)$ such that for all $s > 0$ it holds*

$$\mathcal{F}\big(g_0(s)\big) = \int_0^\infty \frac{e^{-s(1+x)}}{1+x} t_0(dx), \tag{38}$$

$$\mathcal{F}\big(g_\infty(s)\big) = \int_0^\infty \frac{e^{-s(1+x)}}{1+x} t_\infty(dx), \tag{39}$$

$$h_0(s) = \int_{[0,a-1)} \frac{e^{-s(1+x)}}{1+x} t_0(dx). \tag{40}$$

Moreover, the equation

$$\mathcal{F}\big(g_\infty(as)\big) = \mathcal{F}\big(g_0(s)\big) - h_0(s), \qquad s > 0, \tag{41}$$

is equivalent to the relation

$$t_\infty = (1/a)\, t_0 \circ \phi \tag{42}$$

where the linear function $\phi : [0, \infty) \to [0, \infty)$ *is defined as* $\phi(x) = a(1 + x) - 1$.

Remark 5.3. *Combining (H2) and (H3) in Definition 5.1, it follows that the map* $\mathcal{F}$ *is strictly increasing on* $[0, 1)$. *In particular, Equation* (41) *univocally determines* g_∞ *knowing* g_0 *and* h_0 *on* $(0, \infty)$, *and similarly Equations* (38) *and* (39) *univocally determine* g_∞ *and* g_0 *knowing* t_∞ *and* t_0, *respectively.*

We divide the proof of the above theorem in different steps.

Lemma 5.4. *Let* Z *be a random variable such that* $Z \geq 1$ *and define* $g(s) = \mathbb{E}[e^{-sZ}]$, $s \geq 0$. *Let* $w : (0, \infty) \to \mathbb{R}$ *be the unique function such that*

$$\mathcal{F}\big(g(s)\big) = \int_s^\infty du\, e^{-u} w(u), \qquad s > 0, \tag{43}$$

i.e.

$$w(s) := -e^s \mathcal{F}'(g(s)) g'(s), \qquad s > 0. \tag{44}$$

Then the function w *is completely monotone*[3]. *In particular, there exists a unique Radon measure* $t(dx)$ *on* $[0, \infty)$ *(not necessarily of finite total mass) such that*

$$w(s) = \int_0^\infty e^{-sx} t(dx), \qquad s > 0, \tag{45}$$

and therefore

$$\mathcal{F}\big(g(s)\big) = \int_0^\infty \frac{e^{-s(1+x)}}{1+x} t(dx), \qquad s > 0. \tag{46}$$

Moreover, the above identity (46) *univocally determines* $t(dx)$.

Proof. The last statement follows from the inversion formula of the Laplace transform. For the rest, the proof is similar to the proof of Lemma 5.1 in [FMRT0]. The only slight difference is in the following argument. By condition (H2) and since $g(s) \in (0, 1)$ for $s > 0$, we can write $w = f \sum_{k=0}^\infty c_k g^k$, $f = -e^s g'(s)$. Since $c_k \geq 0$ for all $k \geq 0$ and since the product and the sum of completely monotone functions is again completely monotone (cf. [F]) we get that $\sum_{k=0}^\infty c_k g^k$ is completely monotone. The rest of the proof is as in [FMRT0]. $\square$

Lemma 5.5. *Let* Z *be a random variable such that* $Z \geq 1$ *and let* $g(s)$ *be its Laplace transform. Let* t *be the unique Radon measure on* $[0, \infty)$ *satisfying* (46) *and call* $m(dx)$ *the Radon measure with support in* $[1, \infty)$ *such that*

$$m(A) = \int_0^\infty \frac{\mathbb{1}_{1+x \in A}}{1+x} t(dx). \tag{47}$$

For each $k \geq 1$, *consider the convolution measure* $m^{(k)}$ *with support in* $[k, \infty)$ *defined as*

$$m^{(k)}(A) = \int_1^\infty m(dx_1) \int_1^\infty m(dx_2) \cdots \int_1^\infty m(dx_k) \mathbb{1}_{x_1 + x_2 + \cdots + x_k \in A}. \tag{48}$$

[3]Recall that a function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotone if it is $\mathcal{C}^\infty$ and if for any integer k, $(-1)^k f^{(k)} \geq 0$.

Then the law of Z is given by the measure $m_* := \sum_{k=1}^\infty r_k m^{(k)}$, where the coefficients r_k are determined by the series expansion $\mathcal{R}(x) = \sum_{k=1}^\infty r_k x^k$ of the function $\mathcal{R}$ around 0 (recall condition (H3) in Definition 5.1). In particular

$$\mathbb{E}\big[e^{-sZ}; Z < a\big] = \int_{[0,a-1)} \frac{e^{-s(1+x)}}{1+x} t(dx), \qquad s \geq 0. \tag{49}$$

We point out that, given a bounded Borel set A, since $m^{(k)}$ has support in $[k, \infty)$, the series $m_*(A) = \sum_{k=1}^\infty r_k m^{(k)}(A)$ is a finite sum.

Proof. The proof is a generalization of the proof of Lemma 5.2 in [FMRT0]. It reveals the fundamental structure behind the transformation introduced in [FMRT0].

By definition of $m(dx)$ and by (46) we can write

$$\mathcal{F}\big(g(s)\big) = \int_0^\infty \frac{e^{-s(1+x)}}{1+x} t(dx) = \int_0^\infty e^{-sx} m(dx), \qquad s > 0. \tag{50}$$

Since $\lim_{s\to\infty} g(s) = 0$, by (H3) we conclude that $\mathcal{F}(g(s))$ goes to zero as s goes to ∞. In particular, by (H3), for s large enough we can invert (50) and use the analytic expansion of $\mathcal{R}$ getting

$$g(s) = \mathcal{R}\left(\int_0^\infty e^{-sx} m(dx)\right) = \sum_{k=1}^\infty r_k \int_0^k e^{-sx} m^{(k)}(dx), \qquad s \text{ large}, \tag{51}$$

where, in the last equality, we used that $\left(\int_0^\infty e^{-sx} m(dx)\right)^k = \int_0^\infty e^{-sx} m^{(k)}(dx)$. From now on s has to be thought large. We can rewrite the right hand side of (51) as $\sum_{k=1}^\infty \left(\sum_{j=k}^\infty a_{k,j}\right)$, where $a_{k,j} = r_k \int_{I_j} e^{-sx} m^{(k)}(dx)$ and $I_j = [j, j+1)$ for $j \geq 1$. Due to the analytic expansion of $\mathcal{R}(x)$ around 0, we have that $\sum_{k=1}^\infty |r_k x^k| < \infty$ for $|x|$ small, hence we can write

$$\sum_{k=1}^\infty \sum_{j=1}^\infty |a_{j,k}| = \sum_{k=1}^\infty |r_k| \int_0^\infty e^{-sx} m^{(k)}(dx) = \sum_{k=1}^\infty |r_k| \Big(\int_0^\infty e^{-sx} m(dx)\Big)^k < \infty,$$

thus implying that in the series $\sum_{k=1}^\infty \left(\sum_{j=k}^\infty a_{k,j}\right)$ we can indeed arrange the terms as we prefer. In particular, we can invert k and j getting

$$g(s) = \sum_{j=1}^\infty \Big(\sum_{k=1}^j a_{k,j}\Big) = \sum_{j=1}^\infty \int_{I_j} e^{-sx} m_*(dx) = \int_{[0,\infty)} e^{-sx} m_*(dx).$$

Let us write $m_* = m_*^{(+)} - m_*^{(-)}$ as the sum of non-negative measures with disjoint supports [H]. Writing p_Z for the law of Z, we then have that the Laplace transforms of $p_Z + m_*^{(-)}$ and $m_*^{(+)}$ are identical for s large. By Theorem 1a in Section XIII.1 [F] we conclude that $p_Z + m_*^{(-)} = m_*^{(+)}$. Since $m_*^{(-)}$ and $m_*^{(+)}$ have disjoint supports, $m_*^{(-)}$ must be zero and therefore $m_* = m_*^{(+)}$ is a non-negative measure.

To conclude the proof it remains to check (49). It is enough to prove the thesis for $s > 0$, since the case $s = 0$ follows by monotonicity. To this aim we observe that, since $m^{(k)}$ has support contained in $[k, \infty)$ and since $r_1 = \mathcal{R}'(0) = 1$ by (H3) , the measure m_* equals m on $[1, 2)$. Since $a \leq 2$ and using the definition of the measure

m given by (47), we obtain that

$$\mathbb{E}\big[e^{-sZ}; Z < a\big] = \int_{[1,a)} e^{-sx} p_Z(dx) = \int_{[1,a)} e^{-sx} m(dx) = \int_{[0,a-1)} \frac{e^{-s(1+x)}}{1+x} t(dx).$$

This concludes the proof of (49). □

We are now in position to prove Theorem 5.2.

Proof of Theorem 5.2. Observe that Equations (38) and (39) follow from Lemma 5.4, and that Equation (40) follows from (49) in Lemma 5.5.

To prove the last statement we write $\rho(dx)$ for the measure in the r.h.s. of (42). Using that $a[\phi^{-1}(x) + 1] = 1 + x$, we obtain for $s \geq 0$ that

$$\int_0^\infty \frac{e^{-as(1+x)}}{1+x} \rho(dx) = a^{-1} \int_{[\phi(0),\infty)} \frac{e^{-s(1+x)}}{a^{-1}(1+x)} t_0(dx) = \int_{[a-1,\infty)} \frac{e^{-s(1+x)}}{1+x} t_0(dx).$$

Using also (38), (39), (40) we conclude that Equation (41) is equivalent to

$$\int_0^\infty \frac{e^{-as(1+x)}}{1+x} t_\infty(dx) = \int_0^\infty \frac{e^{-as(1+x)}}{1+x} \rho(dx), \qquad \forall s > 0. \tag{52}$$

Thinking the above integrals as Laplace transforms of suitable non-negative measures in the variables as, by Theorem 1a in Section XIII.1 in [F] we conclude that (52) is equivalent to the identity $t_\infty = \rho$. □

6. Asymptotic of the interval law for HCP: Proof of Theorem 2.12

The key result of this section is Theorem 6.1 which in turn allows to prove easily Theorem 2.12. Theorem 6.1 is proved by using the recursive identities for the OCP process established in Section 4 and our extension of the transformation of [FMRT0] derived in the previous section.

Let us start by recalling the notation of Theorem 2.12 which will be used throughout this section and by giving a few more definitions. We let μ be a probability measure on $[d^{(1)}, \infty) = [1, \infty)$ and consider the HCP such that $\xi^{(1)}(0)$ has law of the form $\mathrm{Ren}(\nu, \mu)$, $\mathrm{Ren}(\mu)$ or $\mathrm{Ren}_{\mathbb{Z}}(\mu)$ (ν being a probability measure on $\mathbb{R}$). Call $\mu^{(n)}$ the interval law of $\xi^{(n)}(0)$, *i.e.* at the beginning of epoch n and let $X^{(n)}$ be a generic random variable with law $\mu^{(n)}$ and $Z^{(n)}$ be the rescaled variable $Z^{(n)} = X^{(n)}/d^{(n)}$. Finally, for any $n \geq 1$ and $s \geq 0$ set

$$g^{(n)}(s) := \mathbb{E}\big(e^{-sZ^{(n)}}\big), \qquad h^{(n)}(s) := \mathbb{E}\big(e^{-sZ^{(n)}} \mathbb{1}_{1 \leq Z^{(n)} < a_n}\big) \tag{53}$$

with

$$a_n := d^{(n+1)}/d^{(n)}. \tag{54}$$

Note that $\mu = \mu_1$ and $g^{(1)}(s) = g(s) := \int e^{-sx} \mu(dx)$. The following holds

Theorem 6.1. *Let $\mathcal{F}$ be a function satisfying Hypothesis (H) (see Definition 5.1) and assume that for some number κ it holds*

$$\lim_{s \downarrow 0} -s\mathcal{F}'\big(g(s)\big) g'(s) = \kappa \tag{55}$$

and that

$$\mathcal{F}(g^{(n+1)}(a_n s)) = \mathcal{F}(g^{(n)}(s)) - h^{(n)}(s), \qquad n \geq 1,\ s > 0. \tag{56}$$

Then it must be $\kappa \geq 0$. Moreover, the rescaled variable $Z^{(n)}$ weakly converges to the random variable $Z^{(\infty)} \equiv Z_\kappa^{(\infty)}$ whose Laplace transform $g_\kappa^{(\infty)}$ satisfies

$$\mathcal{F}\big(g_\kappa^{(\infty)}(s)\big) = \kappa \int_1^\infty \frac{e^{-sx}}{x} dx, \qquad s > 0. \tag{57}$$

If $\kappa = 0$ then $Z_\kappa^{(\infty)} = \infty$, while if $\kappa > 0$ then $Z_\kappa^{(\infty)}$ takes value in $[1, \infty)$.

Proof. We first apply Theorem 5.2 getting that, for each $n \geq 1$, there exists a unique measure $t^{(n)}$ on $[0, \infty)$ such that

$$\mathcal{F}\big(g^{(n)}(s)\big) = \int_0^\infty \frac{e^{-s(1+x)}}{1+x} t^{(n)}(dx), \qquad s > 0. \tag{58}$$

Due to (56) and Theorem 5.2 again, for $n \geq 2$ it holds $t^{(n)} = \frac{1}{a_{n-1}} t^{(n-1)} \circ \phi_{n-1}$, with $\phi_{n-1}(x) = a_{n-1}(1+x) - 1$. The recursive identities relying the $t^{(n)}$'s can be explicitly solved, leading to

$$t^{(n)} = \frac{1}{d^{(n)}} t^{(1)} \circ \psi_{n-1}\,, \qquad n \geq 2 \tag{59}$$

with $\psi_{n-1}(x) = d^{(n)}(1+x) - 1$. Defining $U^{(n)}(x) = t^{(n)}([0,x])\mathbb{1}(x \geq 0)$, we get that $dU^{(n)} = t^{(n)}$ and $U^{(n)}(x) = 0$ for $x < 0$. By (59) it holds that

$$U^{(n)}(x) = \frac{1}{d^{(n)}} \Big[U^{(1)}\big(d^{(n)}(1+x) - 1\big) - U^{(1)}\big((d^{(n)} - 1)-\big) \Big], \qquad n \geq 1. \tag{60}$$

Moreover, for each $n \geq 1$, integrating by parts and using that $U^{(n)}(0-) = 0$, we can rewrite the integral in the r.h.s. of (58) as

$$\int_0^\infty \frac{e^{-s(1+x)}}{1+x} t^{(n)}(dx) = \lim_{y \uparrow \infty} \frac{e^{-s(1+y)}}{1+y} U^{(n)}(y) - \int_0^\infty \left(\frac{d}{dx} \left(\frac{e^{-s(1+x)}}{1+x} \right) \right) U^{(n)}(x) dx. \tag{61}$$

We now use the key additional hypothesis (55). Since $g^{(1)}(s) = g(s)$ because $d^{(1)} = 1$, if $w^{(1)}$ denotes the Laplace transform of $t^{(1)}$ (*i.e.* $w^{(1)}(s) = \int_0^\infty e^{-sx} t^{(1)}(dx)$), then (55) together with (44) implies that $\lim_{s \downarrow 0} s\, w^{(1)}(s) = \kappa$. The above limit and the Tauberian Theorem 2 in Section XIII.5 of [F] allow to conclude that

$$\lim_{y \uparrow \infty} \frac{U^{(1)}(y)}{y} = \kappa. \tag{62}$$

The above limit together with (60) implies that there exists a suitable constant $C > 0$ such that

$$U^{(n)}(x) \leq C(1+x), \qquad n \geq 1,\ x \geq 0. \tag{63}$$

In particular, the limit in the r.h.s. of (61) is zero and

$$\int_0^\infty \frac{e^{-s(1+x)}}{1+x} t^{(n)}(dx) = -\int_0^\infty \left(\frac{d}{dx} \left(\frac{e^{-s(1+x)}}{1+x} \right) \right) U^{(n)}(x) dx, \qquad n \geq 1. \tag{64}$$

By (60), (62) and the fact that $d^{(n)} \to \infty$, we conclude that $\lim_{n \to \infty} U^{(n)}(x) = \kappa x$ for all $x \geq 0$. This limit together with (63) allows us to apply the Dominated

Convergence Theorem, getting that

$$\lim_{n\to\infty}\int_0^\infty \frac{e^{-s(1+x)}}{1+x}t^{(n)}(dx) = -\kappa\int_0^\infty \left(\frac{d}{dx}\left(\frac{e^{-s(1+x)}}{1+x}\right)\right)xdx = \kappa\int_0^\infty \frac{e^{-s(1+x)}}{1+x}\,dx \tag{65}$$

(in the last identity we have simply integrated by parts).

Let us come back to (58). We know the limit of the r.h.s. as $n\to\infty$ by (65). Let us analyze the l.h.s. We claim that, given $s>0$, the sequence $\{g^{(n)}(s)\}_{n\geq 1}$ converges to some number in $[0,e^{-s}]$. Indeed, since $Z_n\geq 1$, it holds $g^{(n)}(s)\in(0,e^{-s}]$. If the sequence was not convergent, by compactness we could find two subsequence $\{n_k\}_{k\geq 1}$ and $\{n_r\}_{r\geq 1}$ such that $\lim_{k\to\infty} g^{(n_k)}(s)<\lim_{r\to\infty} g^{(n_r)}(s)$ and both limits exist and belong to $[0,e^{-s}]$. On the other hand, by hypothesis $(H1)$ and Remark 5.3 the function $\mathcal{F}$ is continuous and strictly increasing on $[0,1)$. Hence,

$$\lim_{k\to\infty}\mathcal{F}\big(g^{(n_k)}(s)\big)=\mathcal{F}\big(\lim_{k\to\infty} g^{(n_k)}(s)\big)<\mathcal{F}\big(\lim_{r\to\infty} g^{(n_r)}(s)\big)=\lim_{r\to\infty}\mathcal{F}\big(g^{(n_r)}(s)\big)$$

in contradiction with the fact that the first member and the last member equal the r.h.s. of (65), by (58) and (65).

Since we have proved that for all $s>0$ the sequence $\{g^{(n)}(s)\}_{n\geq 1}$ converges to some number $g_\kappa^{(\infty)}(s)\in[0,e^{-s}]$, using the continuity of $\mathcal{F}$ on $[0,1)$, (58) and (65), we conclude that $g_\kappa^{(\infty)}$ satisfies (57).

Since by Hypothesis (H) the function $\mathcal{F}'$ is positive on $[0,1)$, the limit κ in (55) must be non-negative. Let us first consider the case $\kappa=0$. Then, by (57), the fact that $\mathcal{F}$ is strictly increasing on $[0,1)$ and $\mathcal{F}(0)=0$, we conclude that $g^{(\infty)}(s)=0$ for all $s>0$. This implies that the law of the random variable $Z^{(n)}$ weakly converges to δ_∞.

We now consider the case $\kappa>0$. As pointwise limit of decreasing functions, also $g_\kappa^{(\infty)}$ is decreasing on $(0,\infty)$. In particular the limit $\lim_{s\downarrow 0} g_\kappa^{(\infty)}(s)$ exists and belongs to $[0,1]$. Let us call z this limit and prove that $z=1$. Suppose by absurd that $z\in[0,1)$. Then, by the continuity of $\mathcal{F}$ on $[0,1)$ and Equation (57), we would have

$$\mathcal{F}(z)=\lim_{s\downarrow 0}\mathcal{F}\big(g_\kappa^{(\infty)}(s)\big)=\lim_{s\downarrow 0}\kappa\int_1^\infty \frac{e^{-sx}}{1+x}dx=\infty.$$

Since $\mathcal{F}$ takes finite value on $[0,1)$ it cannot be $\mathcal{F}(z)=\infty$, thus implying that $z=1$. In conclusion we have proved that $\lim_{s\downarrow 0} g_\kappa^{(\infty)}(s)=1$. Then, by Theorem 2 in Section XIII.1 of [F], we conclude that $g_\kappa^{(\infty)}$ is the Laplace transform of some non-negative (finite) random variable $Z_\kappa^{(\infty)}$ and that $Z^{(n)}$ weakly converges to $Z_\kappa^{(\infty)}$. The fact that $Z_\kappa^{(\infty)}\geq 1$ a.s. follows from the fact $Z^{(n)}\geq 1$ for all $n\geq 1$. $\square$

Proof of Theorem 2.12. Thanks to Theorem 2.6 and the discussion before Definition 5.1, the Laplace transforms of the rescaled variables $Z^{(n)}$ satisfy

$$\mathcal{F}(g^{(n+1)}(a_n s))=\mathcal{F}(g^{(n)}(s))-h^{(n)}(s)\qquad\forall n\geq 1,\quad\forall s>0,$$

where

$$\mathcal{F}(x)=\begin{cases}-\ln(1-x) & \text{in case } (i),\\ \frac{\gamma+1}{\gamma+2}\ln\frac{1+\frac{x}{\gamma+1}}{1-x} & \text{in case } (ii),\end{cases}$$

respectively. We have already observed, that in both cases $\mathcal{F}$ satisfies the Hypothesis (H). Computing $\mathcal{F}'$ we get

$$\lim_{s\downarrow 0} -s\mathcal{F}'(g(s))g'(s) = \begin{cases} -\lim_{s\downarrow 0} \frac{sg'(s)}{1-g(s)} & \text{in case } (i), \\ -\lim_{s\downarrow 0} \frac{\gamma+1}{\gamma+2}\left(\frac{sg'(s)}{\gamma+1+g(s)} + \frac{sg'(s)}{1-g(s)}\right) & \text{in case } (ii). \end{cases}$$

Since we have assumed the limit (22) and since $1 - g(s) = o(1)$ for s small, it must be $\lim_{s\downarrow 0} sg'(s) = 0$. This last observation allows to conclude that

$$\lim_{s\downarrow 0} -s\mathcal{F}'(g(s))g'(s) = \begin{cases} c_0 & \text{in case } (i) \\ \frac{\gamma+1}{\gamma+2}c_0 & \text{in case } (ii). \end{cases} \tag{66}$$

At this point Theorem 2.12 is an immediate consequence of Theorem 6.1 and the computation of $\mathcal{R} = \mathcal{F}^{-1}$ given in (36). □

7. Asymptotic of the first point law: proof of Theorem 2.15

In this section we prove Theorem 2.15. While in the derivation of Theorem 2.12 we have tried to keep the discussion at a general and abstract level in order to catch the fundamental structure of the transformation introduced in [FMRT0] and therefore explain the similar asymptotics of very different HCP's, we restrict here to the special cases mentioned in Theorem 2.15. Indeed, as the reader will see, the proof goes through estimates which are very model–dependent.

Proof of Theorem 2.15. Case (i) has been solved in [FMRT0, Theorem 2.24]. Hence we focus on case (ii). Without loss of generality we can restrict to the case $\nu = \delta_0$, *i.e.* when the HCP starts with $\xi^{(1)}(0)$ having law $\text{Ren}(\delta_0, \mu)$, μ being a probability measure on $[d^{(1)}, \infty) = [1, \infty)$. Indeed, in the general case $X_0^{(n)}$ can be expresses as $V + \bar{X}_0^{(n)}$, where $\bar{X}_0^{(n)}$ is the first point in $\xi_0^{(n)}$ for the above HCP starting with distribution $\text{Ren}(\delta_0, \mu)$, while V is a random variable with law ν independent from $\bar{X}_0^{(n)}$. Since $d^{(n)} \to \infty$, when taking the rescaled random variable $Y^{(n)} = X_0^{(n)}/d^{(n)}$ the effect of the random translation V disappears as $n \to \infty$. From Lemma 2.5 we know that the configuration $\xi^{(n)}(0)$ at the beginning of epoch n has law $\text{Ren}\big(\nu^{(n)}, \mu^{(n)}\big)$. As in the previous section $X^{(n)}$ will be a random variable with law $\mu^{(n)}$ and $Z^{(n)}$ the rescaled random variable $Z^{(n)} = X^{(n)}/d^{(n)}$. Moreover, we write $X_0^{(n)}$ for a generic random variable with law $\nu^{(n)}$ and set

$$\ell^{(n)}(s) = \mathbb{E}\Big(e^{-sY^{(n)}}\Big), \qquad s \in \mathbb{R}_+, \qquad Y^{(n)} = X_0^{(n)}/d^{(n)}.$$

Recalling the definitions of $g^{(n)}(s)$, $h^{(n)}(s)$ and a_n in equations (53) and (54), we use formula (12) and Theorem 2.8 (ii) to obtain the recursive equations

$$\ell^{(n+1)}(a_n s) = \ell^{(n)}(s)\sqrt{\frac{1 - g^{(n+1)}(a_n s)^2}{1 - g^{(n)}(s)^2}}\, e^{-h^{(n)}(0)}, \qquad n \geq 1.$$

By iteration, we get

$$\begin{aligned}\ell^{(n)}(s) &= \ell^{(1)}(s/d^{(n)})\sqrt{\frac{1-g^{(n)}(s)^2}{1-g^{(1)}(s/d^{(n)})^2}}\exp\Big\{-\sum_{j=1}^{n-1}h^{(j)}(0)\Big\}\\ &= \ell^{(1)}(s/d^{(n)})\sqrt{\frac{1-g^{(n)}(s)^2}{s}}\sqrt{\frac{s/d^{(n)}}{1-g^{(1)}(s/d^{(n)})^2}}\exp\Big\{\frac{1}{2}\log d^{(n)}-\sum_{j=1}^{n-1}h^{(j)}(0)\Big\}\end{aligned} \tag{67}$$

Since $d^{(n)}\to\infty$, we have $\lim_{n\to\infty}\ell^{(1)}(s/d^{(n)})=1$. By assumption μ has finite mean $\bar\mu=-g'(0)$. Hence, $\frac{s/d^{(n)}}{1-g^{(1)}(s/d^{(n)})^2}$ converges to $1/2\bar\mu$ as $n\to\infty$. Finally, invoking Theorem 2.12, from (67) we get

$$\lim_{n\to\infty}\ell^{(n)}(s)=\frac{1}{\sqrt{2\bar\mu}}\sqrt{\frac{1-\tanh^2\big(\mathrm{Ei}(s)/2\big)}{s}}\lim_{n\to\infty}\exp\Big\{\frac{1}{2}\log d^{(n)}-\sum_{j=1}^{n-1}h^{(j)}(0)\Big\}. \tag{68}$$

In remains to study the last limit in (68). To this aim we come back to the measures $t^{(n)}$. As already observed in the proof of Theorem 6.1 and Theorem 2.12, applying Theorem 5.2 one gets that for each $n\geq 1$ there exists a unique measure $t^{(n)}$ on $[0,\infty)$ satisfying (58) with $\mathcal{F}(x)=\frac{1}{2}\ln\frac{1+x}{1-x}$ for $x\in[0,1)$. Moreover, by Equation (40) it holds

$$h^{(n)}(s)=\int_{[0,a_n-1)}\frac{e^{-s(1+x)}}{1+x}t^{(n)}(dx)$$

and by formula (59) it holds $t^{(n)}=\big(1/d^{(n)}\big)t^{(1)}\circ\psi_{n-1}$ with $\psi_{n-1}(x)=d^{(n)}(1+x)-1$, for all $n\geq 2$. Combining the last identities, from (37) one gets

$$h^{(n)}(s)=\int_{[d^{(n)}-1,\,d^{(n+1)}-1)}\frac{e^{-s(1+x)/d^{(n)}}}{1+x}t^{(1)}(dx),\qquad n\geq 1.$$

The above integral representation implies

$$\sum_{j=1}^{n-1}h^{(j)}(0)=\int_{[0,d^{(n)}-1)}\frac{1}{1+x}t^{(1)}(dx).$$

Equation (27) then follows from Claim 7.1. From this formula one can check that $\lim_{s\to 0}\mathbb{E}(e^{-sY^{(\infty)}})=1$ and $\lim_{s\to\infty}\mathbb{E}(e^{-sY^{(\infty)}})=0$, thus implying $Y^{(\infty)}\in(0,\infty)$. Indeed, it is known that $Ei(x)=-\bar\gamma-\log(x)-\sum_{n=1}^{\infty}\frac{(-x)^n}{n\cdot n!}$ for $x>0$, which, after few computation leads to the limit when $s\to 0$, while for $s\to\infty$, it is enough to observe that $Ei(s)\to 0$ and thus $\tanh(Ei(s)/2)\to 0$.

Finally, we remark that condition (26) is satisfied if μ has finite $(1+\varepsilon)$–moment. Indeed, under this hypothesis it holds $\int_{[1,z]}x^2\mu(dx)\leq z^{1-\varepsilon}\int x^{1+\varepsilon}\mu(dx)\leq Cz^{1-\varepsilon}$ for $\varepsilon\in(0,1)$ and $\int_{[1,\infty)}x^2\mu(dx)<\infty$ if $\varepsilon\geq 1$. □

Claim 7.1.

$$\lim_{z\to\infty}\left(\int_{[0,z-1)}\frac{1}{1+x}t^{(1)}(dx)-\frac{1}{2}\ln z\right)=\frac{1}{2}\log 2+\frac{\bar\gamma}{2}-\frac{1}{2}\ln(\bar\mu) \tag{69}$$

where $\bar\gamma\simeq 0,577$ is the Euler-Mascheroni constant.

Proof of Claim 7.1. First of all we give an explicit formula for the measure $m(dx)$ with support in $[1,\infty)$ such that

$$\int_A m(dx) = \int_{A-1} \frac{1}{1+x} t^{(1)}(dx), \qquad A \subset [1,\infty) \text{ Borel}. \tag{70}$$

Lemma 7.2. *Let $m(dx)$ be the measure defined by (70). Let $\otimes_k \mu$ be the convolution of k copies of the interval law μ. Then*

$$m(A) = \sum_{k=1}^{\infty} \alpha_k [\otimes_k \mu](A), \qquad A \subset [1,\infty) \text{ Borel}$$

where $\alpha_k := \big(1 + (-1)^{k+1}\big)/(2k)$.

Note that, since μ has support in $[1,\infty)$, the probability measure $\otimes_k \mu$ has support in $[k,\infty)$.

Proof. We know that $t^{(1)}$ satisfies (58) with $\mathcal{F}(x) = \operatorname{arctanh}(x)$. Since $g^{(1)}(s) = g(s) := \int e^{-sx}\mu(dx)$, by (70) the identity (58) can be rewritten as $\mathcal{F}(g(s)) = \int_1^\infty e^{-sx} m(dx)$. For s large $g(s)$ goes to zero, hence we can use the analytic expansion of $\mathcal{F}(x)$ around zero (recall that $\operatorname{arctanh}(x) = 1/2 \ln \frac{1+x}{1-x}$ and use (35) with $\gamma = 0$) getting

$$\sum_{k=1}^{\infty} \alpha_k g(s)^k = \int_1^\infty e^{-sx} m(dx), \qquad s \text{ large}.$$

Since $g(s)^k = \int e^{-sx} [\otimes_k \mu](dx)$, the above equation can be written as

$$\sum_{k=1}^{\infty} \alpha_k \int e^{-sx} [\otimes_k \mu](dx) = \int_1^\infty e^{-sx} m(dx), \qquad s \text{ large}.$$

The thesis then follows from Theorem 1a in [F, Section XIII.1]. □

Let $W_1, W_2, \dots, W_k$ be i.i.d. random variables with common law μ. Then, $\otimes_k \mu$ is the law of $W_1 + W_2 + \cdots + W_k$. Due to the above lemma and since $W_i \ge 1$ a.s., we can write

$$\int_{[0,z-1)} \frac{t^{(1)}(dx)}{1+x} = \int_{[1,z)} m(dx) = \sum_{k=1}^{\lfloor z \rfloor} \alpha_k \mathbb{P}(W_1 + \cdots + W_k \le z) \tag{71}$$

where $\lfloor z \rfloor$ denotes the integer part of z.

Recall that $\bar\mu := \int x \mu(dx) = \mathbb{E}(W_i) \ge 1$. If $\bar\mu = 1$ then $\mu = \delta_1$ and, as the reader can check, the arguments below become trivial. Hence, we assume that $\bar\mu > 1$.

Given $z > 1$ we define $\tilde W_i := W_i \mathbb{1}(W_i \le z)$ and $\bar\mu(z) := \mathbb{E}(\tilde W_i) = \mathbb{E}(W_i; W_i \le z)$. We can estimate the variance of $\tilde W_i$ as

$$\operatorname{Var}(\tilde W_i) \le \mathbb{E}(\tilde W_i^2) = \mathbb{E}(W_i^2; W_i \le z). \tag{72}$$

Fix $\varepsilon > 0$. We deal separately with the case (i) $k \le \frac{z}{\bar\mu}(1-\varepsilon)$ and (ii) $k \ge \frac{z}{\bar\mu}(1+\varepsilon)$.

• Case (i). Since $\lim_{z\to\infty} \bar\mu(z) = \bar\mu$, this implies that there exists $z(\varepsilon)$ large enough and independent from k such that for $z \ge z(\varepsilon)$ it holds

$$k\bar\mu(z) < z \qquad \text{and} \qquad \left(\frac{z - k\bar\mu}{z - k\bar\mu(z)}\right)^2 \le 4. \tag{73}$$

Therefore for $z \geq z(\varepsilon)$ thanks to (73) we can use the Markov inequality to obtain

$$\begin{aligned}\mathbb{P}(W_1 + \cdots + W_k > z) &\leq \mathbb{P}(\exists i \leq k \,:\, W_i \neq \tilde{W}_i) + \mathbb{P}(\tilde{W}_1 + \cdots + \tilde{W}_k > z) \\ &\leq k\mathbb{P}(W_1 > z) + \mathbb{P}\Big(\frac{\tilde{W}_1 + \cdots + \tilde{W}_k}{k} - \bar{\mu}(z) > \frac{z - k\bar{\mu}(z)}{k}\Big) \\ &\leq k\mathbb{P}(W_1 > z) + \frac{k\,\mathrm{Var}(\tilde{W}_1)}{(z - k\bar{\mu}(z))^2}.\end{aligned} \tag{74}$$

Then, combining (72), (73) and (74) we get for $z \geq z(\varepsilon)$

$$\begin{aligned}\alpha_k \mathbb{P}(W_1 + \cdots + W_k > z) &\leq \mathbb{P}(W_1 > z) + \frac{\mathbb{E}(W_1^2; W_1 \leq z)}{(z - k\bar{\mu}(z))^2} \\ &\leq \mathbb{P}(W_1 > z) + 4\frac{\mathbb{E}(W_1^2; W_1 \leq z)}{(z - k\bar{\mu})^2}.\end{aligned} \tag{75}$$

• Case (ii) By similar arguments one can prove that there exists $\bar{z}(\varepsilon)$ such that for $z \geq \bar{z}(\varepsilon)$ it holds

$$\alpha_k \mathbb{P}(W_1 + \cdots + W_k \leq z) \leq \mathbb{P}(W_1 > z) + 4\frac{\mathbb{E}(W_1^2; W_1 \leq z)}{(z - k\bar{\mu})^2}. \tag{76}$$

At this point we get

$$\sum_{k=1}^{\lfloor \frac{z}{\bar{\mu}}(1-\varepsilon)\rfloor} \alpha_k + \mathcal{E}_1 \leq \sum_{k=1}^{\lfloor z \rfloor} \alpha_k \mathbb{P}(W_1 + \cdots + W_k \leq z) \leq \sum_{k=1}^{\lfloor \frac{z}{\bar{\mu}}(1+\varepsilon)\rfloor} \alpha_k + \mathcal{E}_2 \tag{77}$$

where the error $\mathcal{E}_1$ can be bounded via (75) as

$$\begin{aligned}|\mathcal{E}_1| &\leq \sum_{k=1}^{\lfloor \frac{z}{\bar{\mu}}(1-\varepsilon)\rfloor} \Big(\mathbb{P}(W_1 > z) + 4\frac{\mathbb{E}(W_1^2; W_1 \leq z)}{(z - k\bar{\mu})^2}\Big) \\ &\leq z\mathbb{P}(W_1 > z) + C\mathbb{E}(W_1^2; W_1 \leq z) \int_{\varepsilon z}^{z} \frac{1}{x^2} dx \leq z\mathbb{P}(W_1 > z) + \frac{C'}{\varepsilon z}\mathbb{E}(W_1^2; W_1 \leq z),\end{aligned}$$

and similarly the error $\mathcal{E}_2$ can be bounded via (76) as

$$|\mathcal{E}_2| \leq z\mathbb{P}(W_1 > z) + \frac{C'}{\varepsilon z}\mathbb{E}(W_1^2; W_1 \leq z).$$

The bound $\bar{\mu} = \mathbb{E}(W_1) < \infty$ trivially implies that $\lim_{z\to\infty} z\mathbb{P}(W_1 > z) = 0$. This observation, together with the hypothesis (26), assures that for any fixed $\varepsilon > 0$ it holds

$$\lim_{z\to\infty} \mathcal{E}_1 = \lim_{z\to\infty} \mathcal{E}_2 = 0. \tag{78}$$

We point out that the above estimates follow closely the arguments used to prove the weak LLN. If μ has finite variance, exactly as in the proof of the LLN, the truncation $\tilde{W}_i$ would be unnecessary and a direct application of the Markov inequality would allow to estimate $\mathcal{E}_1, \mathcal{E}_2$.

It remains to study the behavior of the series $\sum_{k=1}^{n} \alpha_k$ for n integer. It is known that $\sum_{k=1}^{n} \frac{1}{k} = \log n + \bar{\gamma} + o(1)$, where $\bar{\gamma}$ is Euler-Mascheroni constant. Assume that

n is even and $n = 2p$. Then,

$$\phi(n) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{p} \frac{1}{k} = \frac{1}{2}\log 2 + \frac{1}{2}\log n + \frac{\bar{\gamma}}{2} + o(1).$$

For n odd, one obtains a similar expression. Hence,we conclude that

$$\lim_{z\to\infty} \left(\sum_{k=1}^{\lfloor z \rfloor} \alpha_k - \frac{1}{2}\ln z \right) = \frac{1}{2}\log 2 + \frac{\bar{\gamma}}{2}. \tag{79}$$

Collecting (71), (77), (78) and (79) we get that

$$C_* + \frac{1}{2}\ln\left(\frac{z(1-\varepsilon)}{\bar{\mu}}\right) - o(1) \le \int_{[0,z-1)} \frac{t^{(1)}(dx)}{1+x} \le C_* + \frac{1}{2}\ln\left(\frac{z(1+\varepsilon)}{\bar{\mu}}\right) + o(1)$$

where $o(1)$ goes to zero as $z \to \infty$ (for any fixed $\varepsilon > 0$) and $C_* = \frac{1}{2}\log 2 + \frac{\bar{\gamma}}{2}$. Hence

$$\left| \int_{[0,z-1)} \frac{t^{(1)}(dx)}{1+x} - \frac{1}{2}\ln(z/\bar{\mu}) - C_* \right| \le C(\varepsilon + o(1)).$$

At this point take first the limit $z \to \infty$ and then the limit $\varepsilon \downarrow 0$, thus concluding the proof of our claim.

□

8. Universal coupling: graphical construction of the dynamics

In this section we describe the universal coupling for the OCP's. The construction is standard and very similar to the one presented in Section 3.1 of [FMRT0]. On the other hand, it will be used in Section 9.1 and is fundamental in order to recover results as Lemma 8.1 and the first part of Proposition 9.4.

Given $\xi \in \mathcal{N}(d_{\min})$, we enumerate its points in increasing order with the rule that the smallest positive one (if it exists) gets the label 1, while the largest non-positive one (if it exists) gets the label 0. We write $N(x, \xi)$ for the integer number labelling the point $x \in \xi$. This allows to enumerate the domains of ξ as follows: a domain $[x, x']$ is said to be the k^{th}-domain if (i) x is finite and $N(x, \xi) = k$, or (ii) $x = -\infty$ and $N(x', \xi) = k + 1$. Recall that if $x = -\infty$, then ξ is unbounded from the left and x' is the smallest number in ξ.

We set $\|\lambda\|_\infty = \sup_{d\in[d_{\min},d_{\max})} \lambda(d)$ where we recall that $\lambda = \lambda_r + \lambda_\ell + \lambda_a$. We consider a probability space $(\Omega, \mathcal{F}, P)$ on which the following random objects are defined and are all independent: the Poisson processes $\mathcal{T}^{(k)} = \{T_m^{(k)} : m \in \mathbb{N}\}$, $\bar{\mathcal{T}}^{(k)} = \{\bar{T}_m^{(k)} : m \in \mathbb{N}\}$ and $\tilde{\mathcal{T}}^{(k)} = \{\tilde{T}_m^{(k)} : m \in \mathbb{N}\}$ of parameter $\|\lambda\|_\infty$, indexed by $k \in \mathbb{Z}$, and the random variables $U_m^{(k)}$, $\bar{U}_m^{(k)}$ and $\tilde{U}_m^{(k)}$, uniformly distributed in $[0, 1]$, indexed by $k \in \mathbb{Z}$ and $m \in \mathbb{N}$. Above, the Poisson processes are described in terms of the jump times $T_m^{(k)}$, $\bar{T}_m^{(k)}$, $\tilde{T}_m^{(k)}$. By discarding a set of P-probability 0, we may assume that

As k_1, k_2, k_3 vary in $\mathbb{Z}$, the sets $\mathcal{T}^{(k_1)}$,$\bar{\mathcal{T}}^{(k_2)}$ and $\tilde{\mathcal{T}}^{(k_3)}$ are locally finite and disjoint. (80)

Next, given $\zeta \in \mathcal{N}(d_{\min})$ and $\omega \in \Omega$, to each domain Δ that belongs to ζ we associate the Poisson process $\mathcal{T}^{(k)}$ if Δ is the k–th domain in ζ. In this case, we write $\mathcal{T}^{(\Delta)}$ instead of $\mathcal{T}^{(k)}$. Similarly we define $\bar{\mathcal{T}}^{(\Delta)}$, $\tilde{\mathcal{T}}^{(\Delta)}$, $U_m^{(\Delta)}$, $\bar{U}_m^{(\Delta)}$ and $\tilde{U}_m^{(\Delta)}$.

The idea behind the construction of the universal coupling is the following: if for example $s = T_m^{(\Delta)}$ for some $m \in \mathbb{N}$ and if the domain Δ is present at time $s-$, then the left extreme of Δ has to be erased at time s if and only if $U_m^{(\Delta)} \leq \lambda_\ell(d)/\|\lambda\|_\infty$, d being the length of the domain Δ. Similarly, the right extreme (or both the extremes) of Δ can be erased at time $s = \bar{T}_m^{(\Delta)}$ (resp. $\tilde{T}_m^{(\Delta)}$). Working with infinite domains, to formalize the above construction one needs some percolation argument as presented below.

We define $\mathcal{W}_t[\omega, \zeta]$ as the set of domains Δ in ζ such that

$$\Big\{ s \in [0,t] : s \in \mathcal{T}^{(\Delta)} \cup \bar{\mathcal{T}}^{(\Delta)} \cup \tilde{\mathcal{T}}^{(\Delta)}, \text{ or } s \in \mathcal{T}^{(\Delta')} \cup \bar{\mathcal{T}}^{(\Delta')} \cup \tilde{\mathcal{T}}^{(\Delta')} \\ \text{for some domain } \Delta' \text{neighbouring } \Delta \Big\} \neq \emptyset. \qquad (81)$$

On $\mathcal{W}_t[\omega, \zeta]$ we define a graph structure putting an edge between domains Δ and Δ' if and only if they are neighbouring in ζ. Since the function λ is bounded from above, we deduce that the set

$$\mathcal{B}(\zeta) := \Big\{ \omega \,:\, \mathcal{W}_t[\omega, \zeta] \text{ has all connected components of finite cardinality } \forall t \geq 0 \Big\}$$

has P–probability equal to 1. Note that the event $\mathcal{B}(\zeta)$ depends on ζ only through the infimum and the supremum of the set $\{N(x,\zeta) \in \mathbb{Z} \,:\, x \in \zeta\}$. By a simple argument based on countability, we conclude that $P(\mathcal{B}) = 1$, where $\mathcal{B}$ is defined as the family of elements $\omega \in \Omega$ satisfying (80) and belonging to $\cap_{\zeta \in \mathcal{N}(d_{\min})} \mathcal{B}(\zeta)$:

$$\mathcal{B} = \bigcap_{\zeta \in \mathcal{N}(d_{\min})} \mathcal{B}(\zeta) \cap \{\omega \in \Omega : \omega \text{ satisfies } (80)\}. \qquad (82)$$

In order to define the path $\{\xi(s)\}_{s \geq 0} := \{\xi^\zeta(s,\omega)\}_{s \geq 0}$ associated to $\zeta \in \mathcal{N}(d_{\min})$ and $\omega \in \Omega$, we first fix a time $t > 0$ and define the path up to time t. If $\omega \notin \mathcal{B}$, then we set

$$\xi(s) = \zeta, \qquad \forall s \in [0,t].$$

If $\omega \in \mathcal{B}$, recall the definition of the graph $\mathcal{W}_t[\omega, \zeta]$. Given a set of domains V we write $\bar{V}$ for the set of the associated extremes, *i.e.* $x \in \bar{V}$ if and only if there exists a domain in V having x as left or right extreme. Moreover, we write $\mathcal{V}_t[\omega, \zeta]$ for the set of all domains in ζ that do not belong to $\mathcal{W}_t[\omega, \zeta]$. We require that

$$\xi(s) \cap \overline{\mathcal{V}_t[\omega, \zeta]} := \overline{\mathcal{V}_t[\omega, \zeta]}, \qquad \forall s \in [0,t], \qquad (83)$$

i.e. up to time t all points in $\overline{\mathcal{V}_t[\omega, \zeta]}$ survive. Let us now fix a cluster $\mathcal{C}$ in the graph $\mathcal{W}_t[\omega, \zeta]$. The path $\big(\xi(s) \cap \bar{\mathcal{C}} \,:\, s \in [0,t]\big)$ is implicitly defined by the following rules (the definition is well posed since $\omega \in \mathcal{B}$). If $s \in [0,t]$ equals $T_m^{(\Delta)}$ with $\Delta = [x, x'] \in \mathcal{C}$ and $x, x' \in \xi(s-)$, then the ring at time $T_m^{(\Delta)}$ is called *legal* if

$$U_m^{(\Delta)} \leq \frac{\lambda_\ell(x' - x)}{\|\lambda\|_\infty} \qquad (84)$$

and in this case we set $\xi(s) \cap \bar{\mathcal{C}} := (\xi(s-) \cap \bar{\mathcal{C}}) \setminus \{x\}$, otherwise we set $\xi(s) \cap \bar{\mathcal{C}} = \xi(s-) \cap \bar{\mathcal{C}}$. In the first case we say that x is erased and that the domain $[x, x']$ has incorporated the domain on its left. Similarly, if $s \in [0,t]$ equals $\bar{T}_m^{(\Delta)}$ with

$\Delta = [x, x'] \in \mathcal{C}$ and $x, x' \in \xi(s-)$, then the ring at time $\bar{T}_m^{(\Delta)}$ is called *legal* if

$$\bar{U}_m^{(\Delta)} \leq \frac{\lambda_r(x'-x)}{\|\lambda\|_\infty} \tag{85}$$

and in this case we set $\xi(s) \cap \bar{\mathcal{C}} := (\xi(s-) \cap \bar{\mathcal{C}}) \setminus \{x'\}$, otherwise we set $\xi(s) \cap \bar{\mathcal{C}} = \xi(s-) \cap \bar{\mathcal{C}}$. Again, in the first case we say that x' is erased and that the domain $[x, x']$ has incorporated the domain on its right. Finally, if $s \in [0, t]$ equals $\tilde{T}_m^{(\Delta)}$ with $\Delta = [x, x'] \in \mathcal{C}$ and $x, x' \in \xi(s-)$, then the ring at time $\tilde{T}_m^{(\Delta)}$ is called *legal* if

$$\tilde{U}_m^{(\Delta)} \leq \frac{\lambda_a(x'-x)}{\|\lambda\|_\infty} \tag{86}$$

and in this case we set $\xi(s) \cap \bar{\mathcal{C}} := (\xi(s-) \cap \bar{\mathcal{C}}) \setminus \{x, x'\}$, otherwise we set $\xi(s) \cap \bar{\mathcal{C}} = \xi(s-) \cap \bar{\mathcal{C}}$. Again, in the first case we say that x and x' are erased and that the domain $[x, x']$ has incorporated both the domain on its right on its left.

We point out that $\bar{\mathcal{C}} \cap \bar{\mathcal{C}}' = \emptyset$ if $\mathcal{C}$ and $\mathcal{C}'$ are distinct clusters in $\mathcal{W}_t[\omega, \zeta]$. On the other hand, it could be $\bar{\mathcal{C}} \cap \overline{\mathcal{V}_t[\omega, \zeta]} \neq \emptyset$. Let x a point in the intersection and suppose for example that $[a, x] \in \mathcal{C}$ while $[x, b] \in \mathcal{V}_t[\omega, \zeta]$. Then, by definition of $\mathcal{W}_t[\omega, \zeta]$, one easily derives that the Poisson processes associated to the domains $[a, x]$ and $[x, b]$ do not intersect $[0, t]$, while at least one of the Poisson processes associated to the domain on the left of $[a, x]$ intersects $[0, t]$. In particular, $x \in \xi(s) \cap \bar{\mathcal{C}}$ for all $s \in [0, t]$, in agreement with (83). The same conclusion is reached if $[a, x] \in \mathcal{V}_t[\omega, \zeta]$ and $[x, b] \in \mathcal{C}$. This allows to conclude that the definition of the path $\{\xi(s)\}_{s \geq 0}$ up to time t is well posed. We point out that this definition is t–dependent. The reader can easily check that, increasing t, the resulting paths coincide on the intersection of their time domains. Joining these paths together we get $\{\xi(s)\}_{s \geq 0}$.

Given a configuration $\zeta \in \mathcal{N}(d_{\min})$, the law of the corresponding random path $\{\xi(s)\}_{s \geq 0}$ is that of the OCP with initial condition ζ. The advantage of the above construction is that all OCP's, obtained by varying the initial configuration, can be realized on the same probability space. Given a probability measure $\mathcal{Q}$ on $\mathcal{N}(d_{\min})$, the OCP with initial distribution $\mathcal{Q}$ can be realized by the random path $\{\xi^{\cdot}(s, \cdot)\}_{s \geq 0}$ defined on the product space $\Omega \times \mathcal{N}(d_{\min})$ endowed with the probability measure $P \times \mathcal{Q}$.

The next result (similar to [S, Lemma 2.2]) is an immediate consequence of the above construction and of the metric defined on $\mathcal{N}(d_{\min})$. We omit the proof.

Lemma 8.1. *For any $(\zeta, \omega) \in \mathcal{N}(d_{\min}) \times \mathcal{B}$, the function $[0, \infty) \ni s \mapsto \xi^\zeta(s, \omega) \in \mathcal{N}(d_{\min})$ is càdlàg. In other words, $\{\xi^\zeta(s, \omega)\}_{s \geq 0}$ belongs to the Skohorod space $D\big([0, \infty), \mathcal{N}(d_{\min})\big)$.*

9. Proof of Theorem 2.9

This section is dedicated to the construction and the analysis of the Markov generator $\mathcal{L}$ of the OCP. We first introduce the Markov semigroup associated to the graphical construction of Section 8 and then introduce the pregenerator $\mathbb{L}$.

If points (domain extremes) belong always to a given countable subset of $\mathbb{R}$ (for example points belong to $\mathbb{Z}$), then one can directly apply the methods developed for interacting particle systems on countable space [L], identifying each domain extreme with a particle. In the general case, we have introduced a lattice structure (see Section 2.2) which strongly simplifies the problem of the Markov generator

from an analytic viewpoint, and allows us to use again the methods described in [L]. Endowing the space $\mathcal{N}$ of locally finite subset of $\mathbb{R}$ of the vague topology, the map $\mathcal{N} \ni \xi \mapsto \xi \cap [a,b] \in \mathcal{N}$ is not continuous, hence the above discretization requires some special care.

9.1. **Markov semigroup and pregenerator.** Given an initial configuration $\xi \in \mathcal{N}(d_{\min})$, define the path $\{\xi(s)\}_{s\geq 0} = \{\xi^\xi(s)\}_{s\geq 0}$ as in Section 8, with $\zeta = \xi$. Note that the dependence on the element $\omega \in \Omega$ is understood. In what follows we will alternatively use the notation $\{\xi^\xi(s,\omega)\}_{s\geq 0}$, or $\{\xi^\xi(s)\}_{s\geq 0}$ or $\{\xi(s)\}_{s\geq 0}$, depending on the context.

Let $\mathbb{P}_\xi$ be the law of the OCP starting from $\xi \in \mathcal{N}(d_{\min})$:

$$\mathbb{P}_\xi(A) = P\left(\left\{\omega \in \Omega : \{\xi(s)\}_{s\geq 0} \in A\right\}\right) \qquad \forall A \subset D([0,\infty), \mathcal{N}(d_{\min})) \text{ Borel.}$$

We write $\mathbb{E}_\xi$ for the corresponding expectation. Then, for any $f \in \mathbb{B}$, we set

$$P_t f(\xi) = \mathbb{E}_\xi\left(f\left(\xi(t)\right)\right) \qquad \forall t \geq 0.$$

Below we shall prove that $(P_t)_{t\geq 0}$ is a Markov semigroup on $\mathbb{B}$ in the sense of the following definition [L]:

Definition 9.1 (Markov semigroup). *A family of linear operator $(S_t)_{t\geq 0}$ on $\mathbb{B}$ is called a Markov semigroup if it is Feller, i.e. $S_t f \in \mathbb{B}$ for all $f \in \mathbb{B}$, and satisfies the following properties:*
(i) $S_0 = \mathbb{1}_{\mathbb{B}}$, the identity operator on $\mathbb{B}$;
(ii) for any $f \in \mathbb{B}$, $\lim_{t\to 0} \|S_t f - f\| = 0$;
(iii) for any $s,t \geq 0$, any $f \in \mathbb{B}$, $S_{t+s} f = S_t(S_s f)$;
(iv) for any $t \geq 0$, $S_t \mathbb{1} = \mathbb{1}$;
(v) for any $f \in \mathbb{B}$, $f \geq 0 \Rightarrow P_t f \geq 0$.

Before moving to the proof of the fact that $(P_t)_{t\geq 0}$ is a Markov semigroup, we need to introduce some operators and to fix some notation.

Given $s > 0$ we consider the operator L_s on $\mathbb{B}$ defined as

$$\begin{aligned} L_s f(\xi) = \sum_{\substack{[x,x+d] \text{ domain} \\ \text{in } \xi\cap(-s,s)}} \Big\{ & \lambda_\ell(d)\big[f(\xi \setminus \{x\}) - f(\xi)\big] + \lambda_r(d)\big[f(\xi \setminus \{x+d\}) - f(\xi)\big] \\ & + \lambda_a(d)\big[f(\xi \setminus \{x, x+d\}) - f(\xi)\big]\Big\}. \end{aligned} \tag{87}$$

Since ξ is locally finite, the r.h.s. is given by a finite sum and therefore is well defined. Given an integer $n \in \mathbb{N}_+$, we define the operator $\mathbb{L}_n$ on $\mathbb{B}$ as

$$\mathbb{L}_n f(\xi) := d_{\min}^{-1} \int_{n d_{\min}}^{(n+1)d_{\min}} L_s f(\xi) ds. \tag{88}$$

Note that, given $\xi \in \mathcal{N}(d_{\min})$, the integrand is a bounded stepwise function with a finite family of jumps. Hence, it is integrable.

Recall the notation at the beginning of Section 4. Given $k \in \mathbb{Z}$ and $\xi \in \mathcal{N}(d_{\min})$, let

$$\begin{aligned} & c_k(\xi) := \mathbb{1}\big(|\xi \cap I_k| = 1\big)\left[\lambda_r(d^\ell_{z_k}) + \lambda_\ell(d^r_{z_k})\right], \\ & c_{k,k'}(\xi) := \mathbb{1}\big(|\xi \cap I_k| = 1\,,\ |\xi \cap I_{k'}| = 1\,,\ |\xi \cap I_r| = 0\ \forall r : k < r < k'\big)\lambda_a\big(z_{k'} - z_k\big), \end{aligned}$$

where for any ξ such that $|\xi \cap I_k| = 1$ we set $z_k := \xi \cap I_k$ (due to the definition of $\mathcal{N}(d_{\min})$ each interval I_k contains at most one point of ξ).

Finally, for any $f \in \mathbb{D}$ (recall (19)), set

$$\mathbb{L}f(\xi) := \sum_{r\in\mathcal{R}} c_r(\xi)\nabla_r f(\xi), \qquad \xi \in \mathcal{N}(d_{\min}). \tag{89}$$

Since the rates $\lambda_\ell, \lambda_r, \lambda_a$ are bounded, for all $f \in \mathbb{D}$ the series in the r.h.s. of (89) is absolutely convergent, hence $\mathbb{L}f(\xi)$ is well defined.

Lemma 9.2. *The following holds:*

(i) For each $n \in \mathbb{N}_+$, $\mathbb{L}_n$ is a bounded operator from $\mathbb{B}$ to $\mathbb{B}$.
(ii) For each $f \in \mathbb{D}$, $\mathbb{L}f \in \mathbb{B}$ and $\mathbb{L}f = \lim_{n\to\infty} \mathbb{L}_n f$. In particular, $\mathbb{L}$ is an operator with domain $\mathcal{D}(\mathbb{L}) := \mathbb{D}$ into $\mathbb{B}$.
(iii) $\mathbb{B}_{\rm loc} \subset \mathbb{D}$ ($\mathbb{B}_{\rm loc}$ being the set of local functions $f \in \mathbb{B}$).

Remark 9.3. *Observe that, for any $f \in \mathbb{B}_{\rm loc}$ and any $\xi \in \mathcal{N}(d_{\min})$, $\mathbb{L}f(\xi)$ equals the r.h.s. of* (3)*. On the other hand, we point out that the operator $\mathbb{L} : \mathbb{B} \supset \mathbb{D} \to \mathbb{B}$ is a* Markov pregenerator *as defined in* [L, Ch. 1, Def. 2.1]*. Indeed, it holds (i) $\mathbb{1} \in \mathbb{D}$, (ii) $\mathbb{D}$ is dense in $\mathbb{B}$ since it contains the subset $\mathbb{B}_{\rm loc}$ which we know by Lemma 3.2 to be dense and finally (iii) if $f \in \mathbb{D}$ and $f(\xi) = \min\{f(\xi') : \xi' \in \mathcal{N}(d_{\min})\}$ then $\mathbb{L}f(\xi) \geq 0$. Due to* [L, Ch. 1, Prop. 2.2]*, these conditions ensure that $\mathbb{L}$ is a Markov pregenerator.*

Proof. Without loss of generality, for simplicity of notation we take $d_{\min} = 1$.

We consider Part (i). Let $\xi_k \to \xi$ in $\mathcal{N}(d_{\min})$. We set $R := \{s \in [n, n+1] : \xi \cap \{-s, s\} = \emptyset\}$. We claim that $L_s f(\xi_k) \to L_s f(\xi)$ for $s \in R$. To this aim we apply Lemma 3.1 (ii). For k large, it holds that $\xi \cap (-s, s)$ and $\xi_k \cap (-s, s)$ have the same finite cardinality N. Writing x_j and $x_j^{(k)}$ for their j–th point (from the left), we can write

$$\begin{aligned} L_s f(\xi) = \sum_{j=1}^{N-1} \Big\{ &\lambda_\ell(x_{j+1} - x_j)\big[f(\xi \setminus \{x_j\}) - f(\xi)\big] \\ &+ \lambda_r(x_{j+1} - x_j)\big[f(\xi \setminus \{x_{j+1}\}) - f(\xi)\big] \\ &+ \lambda_a(x_{j+1} - x_j)\big[f(\xi \setminus \{x_j, x_{j+1}\}) - f(\xi)\big] \Big\} \end{aligned}$$

and a similar expression for $L_s f(\xi_k)$. The thesis then follows from (a) the convergence $x_j \to x_j^{(k)}$ as $k \to \infty$ due to Lemma 3.1 (ii), (b) the continuity of the jump rates, (c) the convergence $\xi_k \setminus \{x_j^{(k)}\} \to \xi \setminus \{x_j\}$, $\xi_k \setminus \{x_{j+1}^{(k)}\} \to \xi \setminus \{x_{j+1}\}$ and $\xi_k \setminus \{x_j^{(k)}, x_{j+1}^{(k)}\} \to \xi \setminus \{x_j, x_{j+1}\}$ as $k \to \infty$ for $j : 1 \leq j < N$, (d) the continuity of f.

We can now prove that $\mathbb{L}_n f$ belongs to $\mathbb{B}$. To this aim it is enough to apply the dominated convergence theorem together with the above claim and the following observations: (a) $R \setminus [n, n+1]$ is finite, (b) due to the definition of $\mathcal{N}(d_{\min})$ the function $L_s f$ has uniform norm bounded by $Cs\|f\|$, C being independent from s.

Let us now prove Part (ii). Since we already now that $\mathbb{L}_n f \in \mathbb{B}$, it is enough to show that $\sup_{\xi\in\mathcal{N}(d_{\min})} |\mathbb{L}f(\xi) - \mathbb{L}_n f(\xi)|$ converges to zero as $n \to \infty$. By the boundedness of the rates it holds

$$|\mathbb{L}f(\xi) - \mathbb{L}_n f(\xi)| \leq C \sum_{\substack{r\in\mathcal{R} \\ \mathrm{supp}(r)\not\subset[-n,n]}} \Delta_f(r)$$

where the support of r is defined as $\text{supp}(r) = k$ if $r = k$ and $\text{supp}(r) = \{k, k'\}$ if $r = (k, k')$. The above estimate and the fact that $f \in \mathbb{D}$ allow to conclude.

Part *(iii)* is obvious. $\square$

Proposition 9.4. *The family of linear operators $(P_t)_{t\geq 0}$ is a Markov semigroup on $\mathbb{B}$ given by contraction maps (i.e. $\|P_t f\| \leq \|f\|$ for all $f \in \mathbb{B}$). Moreover, for any $f \in \mathbb{B}_{\text{loc}}$, it holds*

$$\lim_{t\downarrow 0} \left\| \frac{P_t f - f}{t} - \mathbb{L} f \right\| = 0. \tag{90}$$

Proof. We focus on the only point that is not standard, namely the Feller property. The rest is either a direct consequence of the graphical construction, or can be easily derived using the arguments presented in [S, Ch. 2]. Details are left to the reader.

Let us prove the Feller property. Fix $f \in \mathbb{B}$ and $\varepsilon > 0$. Thanks to Lemma 3.2, setting $f_N(\xi) = \int_N^{N+1} f(\xi \cap (-r, r)dr$, we are guaranteed that $f_N \in \mathbb{B}_{\text{loc}}$ and $\lim_{N\to\infty} \|f - f_N\| = 0$. Since $\|P_t f - P_t f_N\|_\infty \leq \|f - f_N\|_\infty$ as functions on $\mathcal{N}(d_{\min})$, approximating f by f_N we conclude that it is enough to show that $P_t f_N \in \mathbb{B}$, or equivalently that $P_t f \in \mathbb{B}$ for any $f \in \mathbb{B}_{\text{loc}}$.

Let us fix $f \in \mathbb{B}_{\text{loc}}$ and suppose that f has support inside $(-N, N)$ for some $N \geq 1$. For simplicity of notation we take $d_{\min} \geq 1$ (the general case is completely similar). Since $\mathcal{N}(d_{\min})$ is compact, f is uniformly continuous. Hence, there exists $\delta_0 > 0$ such that

$$m(\zeta, \eta) < \delta_0 \qquad \Longrightarrow \qquad |f(\zeta) - f(\eta)| \leq \varepsilon. \tag{91}$$

Recall the universal coupling discussed in Section 8 and the notation introduced therein. Depending on ε, we can fix $\gamma > 10$ large enough such that $P(\mathcal{C}) \geq 1 - \varepsilon$ where $\mathcal{C}$ is the event given by the elements $\omega \in \Omega$ for which there exist integers k, k' with $10N \leq k, k' \leq \gamma N$ and

$$[0, t] \cap \left(\mathcal{T}^{(j)} \cup \bar{\mathcal{T}}^{(j)} \cup \tilde{\mathcal{T}}^{(j)} \right) = \emptyset \qquad \forall j = k, k-1, -k', -(k'-1).$$

Given a generic configuration $\zeta \in \mathcal{N}(d_{\min})$, all the points x of $\zeta \cap (-N, N)$ have index $N(x, \zeta)$ belonging to $[-N, N]$ due to our assumption $d_{\min} \geq 1$.

We claim that, if $\omega \in \mathcal{C}$, then the configuration $\xi^\zeta(t)[\omega]$ inside $(-N, N)$ is univocally determined knowing $\mathcal{T}^{(j)}$, $\bar{\mathcal{T}}^{(j)}$, $\tilde{\mathcal{T}}^{(j)}$, $(U_m^{(j)})_{m\geq 0}$, $(\bar{U}_m^{(j)})_{m\geq 0}$, $(\tilde{U}_m^{(j)})_{m\geq 0}$ for $j \in [-\gamma N, \gamma N]$. In order to explain this, suppose for example that ζ is unbounded from the left and from the right. Then, the Poisson processes associated to the domains $[x, y]$ and $[y, z]$ do not have any time inside $[0, t]$, where $N(x, \zeta) = k - 1$, $N(y, \zeta) = k$. In particular, both these domains can be incorporated but cannot incorporate other domains. This implies that the point y survives for all times in $[0, t]$. Hence, whatever has happened on the right of y up to time t has not influenced the dynamics on the left of y. The same argument holds observing the domains $[x', y']$ and $[y', z']$ with $N(x', \zeta) = -k'$, $N(y', \zeta) = -(k' - 1)$. If ζ is bounded from the left of from the right, the proof of our claim is even simpler.

Due to the above claim, for each ζ it holds

$$\xi^\zeta(t)[\omega] \cap (-N, N) = \xi^{\bar{\zeta}}(t)[\omega] \cap (-N, N)$$

if $\omega \in \mathcal{C}$ and if $\bar{\zeta}$ is the configuration obtained from ζ by removing all the points $x \in \zeta$ with $|N(x,\zeta)| > \gamma N$. Recalling that f has support in $(-N,N)$, we have

$$\begin{aligned} |P_t f(\zeta) - P_t f(\bar{\zeta})| = \Big| & \int P(d\omega) f\Big(\xi^{\zeta}(t)[\omega] \cap (-N,N)\Big) \\ & - \int P(d\omega) f\Big(\xi^{\bar{\zeta}}(t)[\omega] \cap (-N,N)\Big)\Big| \leq 2P(\mathcal{C}^c)\|f\| \leq 2\|f\|\varepsilon. \end{aligned} \tag{92}$$

Fixed now ζ. Let us suppose for simplicity that ζ is unbounded from the left and from the right (the other cases can be treated similarly). Then $\bar{\zeta}$ contains all the points $x \in \zeta$ with index $N(x,\zeta) \in [-\gamma N, \gamma N]$. We have $\bar{\zeta} = \zeta \cap (-a,b)$ for suitable $a, b > 0$. Due to Lemma 3.1, one can prove that there exists $\delta > 0$ (smaller than δ_0, defined in (91)) such that if $\eta \in \mathcal{N}(d_{\min})$ and $m(\zeta,\eta) \leq \delta$ then $\eta \cap (-a,b)$ has the same cardinality of $\zeta \cap (-a,b)$. In particular, $\eta \cap (-a,b)$ is given by all the points x of η with index $N(x,\eta) \in [-\gamma N, \gamma N]$. This implies that for all $\eta \in \mathcal{N}(d_{\min})$ such that $m(\zeta,\eta) \leq \delta$ it holds $\bar{\eta} = \eta \cap (-a,b)$. Fix $\delta_1 > 0$. Taking δ smaller if necessary, we can assume that if $m(\zeta,\eta) \leq \delta$ then any two points $x \in \zeta$ and $x' \in \eta$ with $N(x,\zeta) = N(x',\eta)$ satisfy $|x - x'| \leq \delta_1$.

Since (92) has been obtained for any configuration in $\mathcal{N}(d_{\min})$, we conclude that

$$|P_t f(\zeta) - P_t f(\eta)| \leq 2\|f\|\varepsilon + |P_t f(\zeta \cap (-a,b)) - P_t f(\eta \cap (-a,b)| \qquad \forall \eta : m(\zeta,\eta) \leq \delta. \tag{93}$$

Hence, in order to prove that $\zeta \mapsto P_t f(\zeta)$ is continuous, it remains to prove that $|P_t f(\zeta \cap (-a,b)) - P_t f(\eta \cap (-a,b)|$ is small with ε. Fix an integer L that will be chosen later and η so that $m(\zeta,\eta) < \delta$. Then we decompose the expectation according to the event that the total (random) number X of clock rings inside $(-a,b)$, up to time t, is smaller or larger than L. Namely

$$\begin{aligned} & |P_t f(\zeta \cap (-a,b)) - P_t f(\eta \cap (-a,b)| \\ & \quad \leq |\mathbb{E}(f(\xi^{\zeta \cap (-a,b)}(t)) \mathbb{1}_{X \leq L}) - \mathbb{E}(f(\xi^{\eta \cap (-a,b)}(t)) \mathbb{1}_{X \leq L})| + 2\|f\| P(X \geq L) \end{aligned} \tag{94}$$

where X is the cardinality of the set

$$\Big\{ s \in [0,t] \; : \; s \in \mathcal{T}^{(k)} \cup \bar{\mathcal{T}}^{(k)} \cup \tilde{\mathcal{T}}^{(k)} \text{ for some } k \in \mathbb{Z} \cap [-\gamma N, \gamma N] \Big\}$$

Let $t_1 < \cdots < t_X$ be clock rings in the above set. Consider the first ring t_1. Either this ring is legal/not legal (see (84), (85), (86)) for both processes (*i.e.* the dynamics starting from $\zeta \cap (-a,b)$ and the dynamics starting from $\eta \cap (-a,b)$), or it is legal for one process and not legal for the other one. In the first case one easily sees that $m(\xi^{\zeta \cap (-a,b)}(t_1) \cap (-N,N)), \xi^{\eta \cap (-a,b)}(t_1) \cap (-N,N)) < \delta$ (and thus $m(\xi^{\zeta \cap (-a,b)}(s) \cap (-N,N)), \xi^{\eta \cap (-a,b)}(s) \cap (-N,N)) < \delta$ for any $s \in [0, t_2)$). The second case takes place with probability bounded by

$$c(\delta_1) := \sup_{i = a, \ell, r} \sup_{d, d' \geq 0 : |d - d'| \leq 2\delta_1} \frac{|\lambda_i(d) - \lambda_i(d')|}{\|\lambda\|}$$

By assumption, the jump rates $\lambda_a, \lambda_\ell, \lambda_r$ are continuous functions with support in $[0, d_{\max}]$, hence they are uniformly continuous and thus $\lim_{\delta_1 \downarrow 0} c(\delta_1) = 0$. Iterating

the above argument, we end up with

$$
\begin{aligned}
&|\mathbb{E}(f(\xi^{\zeta\cap(-a,b)}(t))\mathbb{1}_{X\le L}) - \mathbb{E}(f(\xi^{\eta\cap(-a,b)}(t))\mathbb{1}_{X\le L})| \le Lc(\delta_1)\\
&+\mathbb{E}\left(|f(\xi^{\zeta\cap(-a,b)}(t)) - f_N(\xi^{\eta\cap(-a,b)}(t))|\mathbb{1}_{m(\xi^{\zeta\cap(-a,b)}(t)\cap(-N,N)),\xi^{\eta\cap(-a,b)}(t)\cap(-N,N))<\delta}\right)\\
&\qquad\qquad\qquad\qquad\le Lc(\delta_1)+\varepsilon \qquad (95)
\end{aligned}
$$

where in the last line we used (91) (together with the fact that $\delta < \delta_0$).

In remains to estimate the deviation $P(X \ge L)$ with X a Poisson variable of mean $3tM$, where M is the cardinality of $[-\gamma N, \gamma N]\cap\mathbb{Z}$. Since $E(e^X) = \exp\{(e-1)3tM\}$, setting $L = \kappa tM$ by Chebyshev inequality we get

$$
P(X \ge \kappa tM) \le \exp\{3tM(e-1) - \kappa tM\} \le e^{-\kappa tM/2} \qquad (96)
$$

for $\kappa \ge \kappa_0$. Summing up the above estimates (see (93), (94), (95), (96)) we finally get the following. Fixed $\delta_1 > 0$ and $\kappa > \kappa_0$, for δ small enough the bound $m(\zeta,\eta) < \delta$ implies

$$
|P_t f(\zeta) - P_t f(\eta)| \le 2\|f\|\varepsilon + \kappa tMc(\delta_1) + \varepsilon + \|f\|e^{-\kappa tM/2}.
$$

Choosing κ large enough, and then δ_1 small enough amounts to the desired result. □

9.2. **Proof of Theorem 2.9.** By definition, the Markov generator $\mathcal{L} : \mathbb{B} \supset \mathcal{D}(\mathcal{L}) \to \mathbb{B}$, associated to the Markov semigroup $\{P_t : t \ge 0\}$ acting on the space $\mathbb{B}$, has domain $\mathcal{D}(\mathcal{L})$ given by

$$
\mathcal{D}(\mathcal{L}) := \left\{ f \in \mathbb{B} \,:\, \lim_{t\downarrow 0} \frac{P_t f - f}{t} \text{ exists in } \mathbb{B} \right\}.
$$

Moreover, given $f \in \mathcal{D}(\mathcal{L})$, one sets $\mathcal{L}f := \lim_{t\downarrow 0} \frac{P_t f - f}{t}$. We stress that the above limits are thought w.r.t. the uniform norm. In addition, we recall that the space $\mathbb{B}$ depends on the parameter $d_{\min}$, although omitted. Note that, when speaking of Markov generators, we do not follow the definition given in [L, Ch. 1] (even if, invoking the Hille–Yosida Theorem, the two definitions coincide).

Our aim is to prove the following theorem, which corresponds to Theorem 9.5:

Theorem 9.5. *The subspaces $\mathbb{B}_{\rm loc}$ and $\mathbb{D}$ are a core of the Markov generator $\mathcal{L}$, i.e. $\mathcal{L}$ is the closure of the operator $\mathbb{L} : \mathbb{D} \ni f \mapsto \mathbb{L}f \in \mathbb{B}$, and of its restriction to $\mathbb{B}_{\rm loc}$. Moreover, if $f \in \mathbb{D}$, $\mathcal{L}f(\xi)$ equals the absolutely convergent series in the r.h.s. of* (3).

We need some preparation. Our first target is to prove that the image of $\mathbb{1} - \lambda\mathbb{L}$ (where $\mathbb{1}$ is the identity operator) is dense in $\mathbb{B}$ for λ sufficiently small. To this aim, we follow a strategy similar to the one adopted for particle systems in [L, Ch. 1]. Set $\|c\|_\infty := \sup_{r\in\mathcal{R}} \|c_r\|_\infty$ and note that, by boundedness of the rates, $\|c\|_\infty < \infty$.

Lemma 9.6. *Suppose that $f \in \mathbb{D}$ and $f - \lambda\mathbb{L}f = g$ for some $\lambda \ge 0$. Then for any $r \in \mathcal{R}$ it holds*

$$
\Delta_f(r) \le \Delta_g(r) + \lambda \sum_{r'\in\mathcal{R},\, r'\ne r} \gamma(r,r')\Delta_f(r') \qquad (97)
$$

where $\gamma(r,r') := \sup_{\xi\in\mathcal{N}(d_{\min})} \left|c_{r'}(\xi^r) - c_{r'}(\xi)\right|$.

Proof. Fix $\varepsilon > 0$ and a finite subset $\hat{\mathcal{R}} \subset \mathcal{R}$. Take $\xi \in \mathcal{N}(d_{\min})$ such that $\Delta_f(r) \le \varepsilon + |\nabla_r f(\xi)|$. Since the map $\xi \mapsto f(\xi^r)$ can be discontinuous we cannot avoid the error ε (the setting in [L] is different due to continuity). We first consider the case that $|\nabla_r f(\xi)| = \nabla_r f(\xi)$. Then, it holds

$$\Delta_f(r) \le \varepsilon + \nabla_r f(\xi) = \varepsilon + \nabla_r g(\xi) + \lambda \mathbb{L} f(\xi^r) - \lambda \mathbb{L} f(\xi) \le \varepsilon + \Delta_g(r) + \lambda \mathbb{L} f(\xi^r) - \lambda \mathbb{L} f(\xi). \tag{98}$$

Since $\xi^r = \xi \setminus I_r$ ($I_r := I_k \cup I_{k'}$ if $r = (k, k')$), $c_r(\xi^r) = 0$ and $\nabla_r f(\xi) \ge 0$, we have

$$\begin{aligned} \mathbb{L} f(\xi^r) - \mathbb{L} f(\xi) &= \sum_{r' \in \mathcal{R}} \left\{ c_{r'}(\xi^r) \nabla_{r'} f\big(\xi^r\big) - c_{r'}(\xi) \nabla_{r'} f(\xi) \right\} \\ &\le \sum_{r' \in \mathcal{R},\, r' \neq r} \left\{ c_{r'}(\xi^r) \nabla_{r'} f\big(\xi^r\big) - c_{r'}(\xi) \nabla_{r'} f(\xi) \right\}. \end{aligned} \tag{99}$$

By our choice of ξ we can write

$$f\big((\xi^{r'})^r\big) - f\big(\xi^{r'}\big) \le \Delta_f(r) \le \varepsilon + f(\xi^r) - f(\xi),$$

thus implying that $\nabla_{r'} f\big(\xi^r\big) \le \varepsilon + \nabla_{r'} f(\xi)$. In particular, it holds

$$\begin{aligned} c_{r'}(\xi^r) \nabla_{r'} f\big(\xi^r\big) - c_{r'}(\xi) \nabla_{r'} f(\xi) &\le \big[c_{r'}(\xi^r) - c_{r'}(\xi)\big] \nabla_{r'} f(\xi) + \varepsilon \|c\|_\infty \\ &\le \gamma(r, r') \Delta_f(r') + \varepsilon \|c\|_\infty. \end{aligned} \tag{100}$$

On the other hand, we have the trivial bound

$$c_{r'}(\xi^r) \nabla_{r'} f\big(\xi^r\big) - c_{r'}(\xi) \nabla_{r'} f(\xi) \le 2\|c\|_\infty \Delta_f(r'). \tag{101}$$

Combining (98), (99) and using (100) for $r' \in \hat{\mathcal{R}}$ and (101) for $r' \in \mathcal{R} \setminus \hat{\mathcal{R}}$, we get

$$\Delta_f(r) \le \varepsilon + \Delta_g(r) + \lambda \sum_{r' \in \hat{\mathcal{R}} : r' \neq r} \gamma(r, r') \Delta_f(r') + \lambda \varepsilon \|c\|_\infty |\hat{\mathcal{R}}| + 2\lambda \|c\|_\infty \sum_{r' \in \mathcal{R} \setminus \hat{\mathcal{R}}} \Delta_f(r'). \tag{102}$$

It is simple to check, by similar arguments, that the above bound (102) holds also in the case $|\nabla_r f(\xi)| = -\nabla_r f(\xi)$. Note moreover that, since $f \in \mathbb{D}$, the last series in (102) is finite and converges to zero as $\hat{\mathcal{R}} \nearrow \mathcal{R}$. Taking first the limit $\varepsilon \downarrow 0$ and then the limit $\hat{\mathcal{R}} \nearrow \mathcal{R}$ we get the thesis. $\square$

We can finally prove our first target:

Lemma 9.7. *The image $\{f - \lambda \mathbb{L} f \,:\, f \in \mathbb{D}\}$ is dense in $\mathbb{B}$ for $\lambda \ge 0$ small enough.*

Proof. Part of the proof is similar to the proof of [L, Ch. 1, Thm. 3.9]. We give it for completeness. Without loss of generality, for simplicity of notation we take $d_{\min} = 1$. Consider the operator $\mathbb{L}_n$ defined in (88). As already observed in Lemma 9.2, $\mathbb{L}_n$ is a bounded operator $\mathbb{L}_n : \mathbb{B} \to \mathbb{B}$. It is simple to check that $\mathbb{L}_n$ is a Markov pregenerator (see the criterion in Remark 9.3). Being $\mathbb{L}_n$ a bounded Markov pregenerator, the image of $\mathbb{1} - \lambda \mathbb{L}_n$ is the entire space $\mathbb{B}$ for each $\lambda \ge 0$ (see [L, Ch. 1,Prop. 2.8]). Hence, fixed $g \in \mathbb{D}$ we can find $f_n \in \mathbb{B}$ such that

$$f_n - \lambda \mathbb{L}_n f_n = g.$$

Take $s \in (n, n+1)$. Fix $r \in \mathcal{R}$. If $\nabla_r f_n(\xi) \ge 0$ we can bound

$$L_s f_n(\xi^r) - L_s f_n(\xi) \le \sum_{\substack{r' \in \mathcal{R} : r' \neq r, \\ \mathrm{supp}(r') \subset [-n-1, n+1)}} \mathcal{U}\Big(c_{r'}(\xi^r) \nabla_{r'} f_n\big(\xi^r\big) - c_{r'}(\xi) \nabla_{r'} f_n(\xi) \Big)$$

where $\mathcal{U}(x) = x\mathbb{1}_{\{x\geq 0\}}$. Hence, averaging over s, the same estimate holds for $\mathbb{L}_n$ instead of L_s. Using this observation and the same arguments used in the proof of Lemma 9.6, we get

$$\Delta_{f_n}(r) \leq \Delta_g(r) + \lambda \sum_{\substack{r'\in\mathcal{R}: r'\neq r,\\ \mathrm{supp}(r')\subset[-n-1,n+1)}} \gamma(r,r')\Delta_{f_n}(r'). \tag{103}$$

Introduce now the bounded operator $\Gamma : \ell_1(\mathcal{R}) \to \ell_1(\mathcal{R})$ as

$$(\Gamma \underline{x})(r) = \sum_{r'\in\mathcal{R}: r'\neq r} \gamma(r,r')\underline{x}(r'), \qquad \underline{x} \in \ell_1(\mathcal{R}).$$

The operator is bounded since $\gamma(r,r')$ is bounded by $\|c\|_\infty$ and is zero if the supports of r and r' are at distance larger than a suitable constant depending on $d_{\min}$ and $d_{\max}$ only (recall that that rates $\lambda_\ell, \lambda_r, \lambda_a$ are zero when evaluated at $d \geq d_{\max}$). Then, the bound (103) implies that $[\mathbb{1} - \lambda\Gamma]\Delta_{f_n} \leq \Delta_g$. If λ is small enough, the operator $\mathbb{1} - \lambda\Gamma$ can be inverted and therefore we get

$$\Delta_{f_n} \leq [\mathbb{1} - \lambda\Gamma]^{-1}\Delta_g. \tag{104}$$

Let us define $g_n := f_n - \lambda\mathbb{L}f_n$. Then

$$\begin{aligned}\|g - g_n\| = \lambda\|(\mathbb{L} - \mathbb{L}_n)f_n\| &\leq \sum_{\substack{r\in\mathcal{R}:\\ \mathrm{supp}(r)\not\subset(-n,n)}} \|c_r\|_\infty \Delta_{f_n}(r)\\ &\leq \|c\|_\infty \sum_{\substack{r\in\mathcal{R}:\\ \mathrm{supp}(r)\not\subset(-n,n)}} [\mathbb{1} - \lambda\Gamma]^{-1}\Delta_g(r).\end{aligned}$$

Since $[\mathbb{1} - \lambda\Gamma]^{-1}\Delta_g \in \ell_1(\mathcal{R})$, the above bound implies that $\lim_{n\to\infty}\|g - g_n\| = 0$. Recalling that $g \in \mathbb{D}$ and that g_n belongs to the image of $\mathbb{1} - \lambda\mathbb{L}$, we conclude that the image of this last operator is dense in $\mathbb{D}$ and therefore in $\mathbb{B}$. □

As a consequence of the above result and Remark 9.3, we get that the closure $\bar{\mathbb{L}}$ of $\mathbb{L}$ is a Markov generator in the sense of [L, Ch. 1, Def. 2.7] (briefly, we will say that $\mathbb{L}$ is a L–Markov generator).

Lemma 9.8. *If $f \in \mathbb{D}$, then there exists a sequence $f_n \in \mathbb{B}_{\mathrm{loc}}$ such that $f_n \to f$ and $\mathbb{L}f_n \to \mathbb{L}f$ in $\mathbb{B}$.*

Proof. Given n set $f_n(\xi) := \int_n^{n+1} f(\xi\cap(-s,s))ds$. Due to Lemma 3.2, we know that $\|f - f_n\| \to 0$ and $f_n \in \mathbb{B}_{\mathrm{loc}}$. Let us prove that $\|\mathbb{L}f_n - \mathbb{L}f\| \to 0$. To this aim, setting $\xi_s := \xi\cap(-s,s)$ and observing that $(\xi_s)^r = (\xi^r)_s$ for all $r \in \mathcal{R}$, for any integer N we can write

$$\begin{aligned}\Big|\mathbb{L}f(\xi) - \mathbb{L}f_n(\xi)\Big| &= \Big|\int_n^{n+1}\sum_{r\in\mathcal{R}} c_r(\xi)\big(\nabla_r f(\xi) - \nabla_r f(\xi_s)\big)ds\Big|\\ &\leq \Big|\int_n^{n+1}\sum_{\substack{r\in\mathcal{R}:\\ \mathrm{supp}(r)\not\subset[-N,N]}} c_r(\xi)\big(\nabla_r f(\xi) - \nabla_r f(\xi_s)\big)ds\Big|\\ &\quad + \Big|\int_n^{n+1}\sum_{\substack{r\in\mathcal{R}:\\ \mathrm{supp}(r)\subset[-N,N]}} c_r(\xi)\big(\nabla_r f(\xi) - \nabla_r f(\xi_s)\big)ds\Big|\\ &\leq 2\|c\|_\infty \sum_{\substack{r\in\mathcal{R}:\\ \mathrm{supp}(r)\not\subset[-N,N]}} \Delta_f(r) &(105)\\ &\quad + 2\|c\|_\infty |\{r\in\mathcal{R} : \mathrm{supp}(r)\subset[-N,N]\}| \cdot \|f - f_n\|. &(106)\end{aligned}$$

Given $\varepsilon > 0$ we choose N large enough that (105) is smaller than ε (this is possible since $f \in \mathbb{D}$). Afterwards, for n large enough (106) is smaller than ε (recall that $f_n \to f$ in $\mathbb{B}$). Then we conclude that $\|\mathbb{L}f - \mathbb{L}f_n\| \leq 2\varepsilon$ for n large enough. □

We can finally prove Theorem 9.5.

Proof of Theorem 9.5. In Proposition 9.4 we have already showed that $\mathcal{L}f = \mathbb{L}f$ if $f \in \mathbb{B}_{\rm loc}$. As observed after Lemma 9.2, in this case $\mathcal{L}f$ must equal (3). By Lemma 9.8, $\bar{\mathbb{L}}$ is the closure of the restriction of $\mathbb{L}$ to $\mathbb{B}_{\rm loc}$. Hence, $\mathbb{B}_{\rm loc}$ is a core of $\bar{\mathbb{L}}$. By Lemma 9.2 (i), given $f \in \mathbb{D}$ the value $\mathbb{L}f(\xi)$ equals the r.h.s. of (3) which is an absolutely convergent series.

It remains to prove that $\bar{\mathbb{L}} = \mathcal{L}$. Since $\mathbb{L}f = \mathcal{L}f$ for all $f \in \mathbb{B}_{\rm loc}$, Lemma 9.8 and the closure of $\mathcal{L}$ implies that $f \in \mathcal{D}(\mathcal{L})$ and $\mathbb{L}f = \mathcal{L}f$ for all $f \in \mathbb{D}$ (the fact that $\mathcal{L}$ is close is a standard fact: combine Def. 2.1 in [L, Ch. 1] with the Hille–Yosida Theorem as stated in Thm. 2.9 in [L, Ch. 1] leading to the fact that $\mathcal{L}$ is an L–Markov generator, and therefore close). This observation implies that $\mathcal{L}$ is an extension of $\bar{\mathbb{L}}$. It is a general fact that this implies that $\mathcal{L} = \bar{\mathbb{L}}$ (cf. [S, Prop. 3.13] together with the Hille–Yosida Theorem as stated in Thm. 2.9 in [L, Ch. 1]).

□

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